

NONCOMMUTATIVE ALGEBRAIC GEOMETRY II

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Introduction

Using the notion of non-commutative deformation of modules, worked out in [16-17], I gave, in [18], a first, very sketchy, version of a construction of a non-commutative algebraic geometry. In this paper I have worked out some of the ideas of [18]. In particular I have constructed a presheaf of observables defined on the ordered set of subdiagrams of a universe $\underline{\mathcal{C}}$. Moreover, I claim that this construction is, at least in some cases, usefull for the understanding of invariant and moduli theory.

The construction is dependent upon the choice of a reasonable abelian category of objects $\underline{\mathcal{C}}$, the universe. Since the process of generalizing will be clear, I shall assume that we are given a k -algebra A , and that we pick as our universe $\underline{\mathcal{C}}$, the category of right A -modules.

As a model we shall take the classical construction of the scheme $X := \text{Spec}(A)$ when A is a commutative finite type k -algebra, with k algebraically closed. A point of X is a prime ideal \mathfrak{p} of A , or rather the right (or left) A -module A/\mathfrak{p} . A closed point of $\text{Spec}(A)$ is a simple module of A , i.e. the residue field $k(x)$ of a closed point $x \in X$, considered as an A -module. Moreover, X is obviously the moduli space of its closed points, implying that the hull $H(k(x))$ of the deformation functor $\text{Def}_{k(x)}$, is the completion \hat{A}_x of O_X at the point x . The regular functions f of X , i.e. the sections of the structure sheaf O_X , are analytically determined by the family of Taylor series $\hat{f}_x \in \hat{A}_x$ at the different points $x \in X$, therefore by their (right multiplicative) *actions* on the hull of the different deformation functors $\text{Def}_{k(x)}$.

This completion process, and the corresponding identification of a regular function f on X as an operator in the k -vectorspace A as well as in \hat{A}_x for every $x \in X$, is going to replace the localization process of classical scheme theory.

Recall that to recover the affine ring from the scheme

$$(i) \quad (X, O_X)$$

we are dependent upon the Zariski topology on X and on the sheaf property of O_X , both stemming from the process of localization for commutative rings. We find,

$$(ii) \quad A = H^0(X, O_X)$$

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Recall also that for non commutative rings, the localization process functions only for Ore-sets, which are scarce. To obtain a good non commutative theory we shall therefore have to change the notion of *space*, conserving the notion of points and incidences, but (seemingly) leaving out the topology. The notion of *structure sheaf* must therefore be changed.

Let \underline{c} be any reasonable *diagram* of \underline{C} , and let

$$\pi : \underline{C} \rightarrow k - mod.$$

be the obvious forgetful functor. We shall assume, for the rest of this paper, that all objects V_i, V_j of \underline{c} are such that,

$$\dim_k Ext_A^1(V_i, V_j) < \infty$$

It is possible, by introducing topologies, to treat the case where we just assume these dimensions are bounded by \aleph_0 , see [12]. Let $|\underline{c}|$ denote the family of objects of \underline{c} , and assume first that \underline{c} is finite. Let $H(|\underline{c}|) = (H_{i,j})$ be the hull of the non commutative deformation functor of this family of A-modules. To \underline{c} we now associate a subalgebra

$$O^A(\underline{c}, \pi) \subseteq (H_{i,j}(|\underline{c}|) \otimes_k Hom_k(V_i, V_j))$$

of *preobservables*, acting on \underline{c} , and a canonical homomorphism

$$A \xrightarrow{\eta(\underline{c})} O^A(\underline{c}, \pi),$$

such that O is a closure operator, i.e. such that

$$(iii) \quad O \simeq O^O(\underline{c}, \pi)$$

Notice that I shall, abusing the notations, write \otimes where one should have written $\hat{\otimes}$, i.e. when $H(V_i) \in \hat{\underline{a}}_r$ but $H(V_i) \notin \underline{a}_r$ and where we therefore have to work with complete tensor products.

To extend this construction to infinite diagrams, we have to sheafify the O -construction, obtaining for every finite diagram \underline{c} a smaller k-algebra, $\mathcal{O}(\underline{c}, \pi)$ containing the image of η . This new ring of *observables* has good functorial properties and we may easily extend it to infinite diagrams \underline{c} . The final *non commutative structure sheaf* \mathcal{O}_π is a certain quotient of this $\mathcal{O}(-, \pi)$, see section (2). We then proclaim,

Definitions 2.8 and 2.17. A diagram \underline{c} of \underline{C} will be called a *prescheme* for A , if

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}(\pi, \underline{c})$$

is an isomorphism. If this is the case, (\underline{c}, A) is called an *affine prescheme* and we shall refer to A as the *affine ring* of this *prescheme*. The diagram \underline{c} will be called a *scheme* for A , if

$$\eta(\underline{c}) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

is an isomorphism. If this is the case, $(\underline{\mathcal{C}}, A)$ is called an affine scheme and we shall refer to A as the affine ring of this scheme.

In particular, if $\underline{\mathcal{C}}$ is finite, the pair,

$$(iv) \quad (\underline{\mathcal{C}}, \mathcal{O}^A(\underline{\mathcal{C}}, \pi))$$

is, by definition, an affine (non-commutative) prescheme. Thus, (ii) in the commutative scheme theory, is replaced by (iii) in the general case.

Notice that we are now talking about *a scheme for A* , not about *the scheme for A* . In fact it is easily seen that there may be several usefull schemes for an algebra A , depending upon what kind of properties of the algebra one would like to study.

Non-commutative algebraic geometry is concerned with these affine schemes, and their globalizations. The categorical properties of our universe $\underline{\mathcal{C}}$, replace the topology and the classical structure sheaf \mathcal{O}_X is replaced by the \mathcal{O}_π -construction.

As an example, let us consider the 0-dimensional case. If A is a commutative k -algebra of dimension 0, then $A = \bigoplus_{i=1}^r \mathcal{O}_{X, x_i}$ where $X = \text{Spec}(A) = \{x_1, \dots, x_r\}$. The corresponding non-commutative situation is the subject of the paper [17]. Let A be a finite dimensional k -algebra, k algebraically closed, and $\mathcal{V} = \{V_i\}$ the (finite) family of all simple modules. We shall consider each module of this family as a point, and we shall consider the obvious forgetful functor $\pi : A\text{-mod} \rightarrow k\text{-mod}$. The local ring (or the infinitesimal neighbourhood) of a point V_i of \mathcal{V} , the analogue of the completion $\hat{\mathcal{O}}_{X, x}$ of the local ring $\mathcal{O}_{X, x}$ of a closed point $x \in X = \text{Spec}(A)$, is the algebra

$$(v) \quad H(V_i) \otimes_k \text{End}_k(V_i)$$

where $H(V_i)$ is the hull of the deformation functor Def_{V_i} .

The affine ring A is, however, no longer isomorphic to the sum of these local algebras. Here is where the notion of non-commutative deformations enters. Let $H(\mathcal{V}) = (H_{i,j})$ be the hull of the non-commutative deformation functor of the family $\mathcal{V} = \{V_i\}$, see [16], then the *infinitesimal interactions* of the *points* of \mathcal{V} , translates into the components

$$(vi) \quad H_{i,j}(\mathcal{V}) \otimes_k \text{Hom}_k(V_i, V_j), i \neq j$$

of the ring of *observables* $O(\mathcal{V}) := O(\mathcal{V}, \pi)$, see [16]. There is a natural morphism of k -algebras,

$$\eta : A \longrightarrow O(\mathcal{V}) := (H_{i,j}(\mathcal{V}) \otimes_k \text{Hom}_k(V_i, V_j))$$

which, according to the *Generalized Burnside Theorem*, is an isomorphism.

This is the *Serre theorem*, i.e. the analogue of (ii), in the 0-dimensional non-commutative algebraic geometry. And

$$(\mathcal{V} := \{V_i\}_{i=1}^r, A)$$

is the corresponding non commutative affine scheme.

Notice that in the construction of $H(\mathcal{V})$ we only use the structure of the abelian category $\underline{\mathcal{C}}$ (of A -modules) in which we consider our family of objects \mathcal{V} . $H(\mathcal{V})$ is

therefore an invariant of the Morita equivalence class of A . To recover A , i.e. in the construction of the ring of observables, we must also know the dimensions of the different points V_i of the non-commutative scheme $\underline{\mathcal{C}} := \mathcal{V}$, i.e. we must know the forgetful functor π . However, as we have shown in [17], $H(\mathcal{V})$ is Morita equivalent to A !

-Now, to call something a geometry, one should certainly have the possibility of defining some kind of hierarchy among the *geometrical subobjects*, something like a *quiver of incidences*. Given a geometrical subobject we should at least be able to decide which points *sit* on the subvariety. In our case, if $\underline{\mathcal{C}}$ is an affine scheme for A , the morphisms of $\underline{\mathcal{C}}$ correspond to incidences among the points. Moreover, as we have seen in the 0-dimensional case discussed above, there may also be some *infinitesimal* incidences between the points V_i and V_j , corresponding to a k -basis of $Ext_A^1(V_i, V_j)$. And these are essential in the (re)construction of the affine ring of observables.

-To qualify as a geometry, any model should include a *dynamical* element, i.e. either a topology and a differential structure, including vectorfields, or something taking its place. This is, in our case, provided by a differential calculus induced by the deformation theory, see (2.19) and (2.23), where the basic notions are introduced.

-To be taken seriously, any non-commutative algebraic geometry must certainly include the classical algebraic geometry as a special case. To see that our model satisfies this condition, let A be a commutative k -algebra. The points of the affine scheme $\text{Spec}(A)$ may be identified with the members of the family of indecomposable modules $\mathcal{V} = \{A/\mathfrak{p}\}_{\mathfrak{p} \in \text{Spec}(A)}$. We shall consider this family of A -modules together with the obvious canonical morphisms, obtaining a (usually infinite) diagram (really an ordered set) $\underline{\mathcal{C}} = \text{Spec}(A)$, of A -mod. Notice that $\text{Spec}(A)$ as a set, contains the set of closed points, $\text{Simp}(A)$, containing the *simple* A -modules, together with all the irreducible subvarieties of $\text{Spec}(A)$, considered both as points in their own right, and as defining subschemes, with the corresponding points as their *generic points*. This induces a notion of incidence among different points in the *geometry*, just as we have done above. Recall, however, that in classical scheme theory, a scheme is the moduli space of its closed points, but not necessarily of the non-closed points. There is, in fact, a dicotomy between the set of closed points, and the set of non-closed points, between the scheme and its *Hilbert schemes*.

In our non-commutative geometry, the general notion of scheme is an intermediate version, providing us with a set of *points* and incidences, such that all points are on equal footing, see section (2).

These considerations lead us to the diagram of A -modules, $\text{Simp}^*(A)$, consisting of A , the *projective generator*, and all the simple A -modules, together with the obvious incidences.

The imbedding of the classical algebraic geometry (defined on an algebraically closed field k), into the proposed non-commutative algebraic geometry, is taken care of by the following result,

Theorem 3.1. *Let A be any commutative k -algebra of finite type, k algebraically closed. Then the canonical morphism of k -algebras*

$$\eta(\text{Simp}^*(A), \pi) : A \rightarrow \mathcal{O}_\pi(\text{Simp}^*(A))$$

is an isomorphism.

We shall look at invariant theory, in this general setting, and in particular, we shall see that many problems of moduli in algebra, which cannot be treated in the classical framework of schemes, or of ringed spaces, have very satisfactory solutions expressible in the language of this generalized scheme theory, see section (6). We shall also, see section (2), in relation to the problem of defining quotients of a *space* w.r.t. an equivalence relation, discuss the relationship between this non commutative *algebraic geometry*, and the non commutative *geometry* and its *espaces quantiques* of Alain Connes, [7]. It is clear that, although the starting point and the techniques used, are quite different, the basic ideas are closely related.

To get a different view of what we are after, consider the following:

-Given an algebraic object, say a singularity. We know via deformation theory what one should mean by its (infinitesimal) deformations. Consider a versal (flat) family. We are looking for an algebra of operators, called *observables*, with the property that the set of simple representations of this algebra is in one-to-one correspondence with the isomorphism classes of singularities in the given family. Moreover, this algebra of observables should contain all the information about the possible deformations, or changes of *states*, of the algebraic object, including the abrupt changes observed in families for which some discrete invariant jumps.

-This is analogous to the set-up of quantum theory. Replace the singularity with the (platonic?) idea of some reality *out there*, say a fundamental particle, then the state space of quantum theory is a module, or a representation, of a ring of generalized coordinate functions, the *observables*. Notice that if the simple modules of this ring had all been isomorphic to some field of *numbers*, the observables would have been provided with definite values characterizing the possible states of the fundamental particle, as in classical physics. When, however, the algebra of observables is non-commutative, all the simple representations may be of infinite dimensions on the base-field. The numerical trick does not function. The only invariant *values* of an observable is then the eigenvalues. We are therefore left with a new quantized description of the space of realities, in which measurement must be redefined, and time and dynamics rethought.

I hope that the non-commutative algebraic geometry I am proposing may also contribute to a better understanding of this situation.

1. Homological preparations.

*Ext*s and *Hochschild cohomology*. Let k be a (usually algebraically closed) field, and let A be a k -algebra. Denote by $A\text{-mod}$ the category of right A -modules and consider the exact forgetful functor

$$\pi : A\text{-mod} \longrightarrow k\text{-mod}$$

Given two A -modules M and N , we shall always use the identification

$$\sigma^i : \text{Ext}_A^i(M, N) \simeq HH^i(A, \text{Hom}_k(M, N)) \text{ for } i \geq 0$$

If L_* and F_* are A -free resolutions of M and N respectively, and if an element

$$\xi \in \text{Ext}_A^1(M, N)$$

is given in Yoneda form, as

$$\xi = \{\xi_n\} \in \prod_n \text{Hom}_A(L_n, F_{n-1})$$

then $\sigma^1(\xi)$ is gotten as follows. Let σ be a k -linear section of the augmentation morphism

$$\rho : L_0 \longrightarrow M$$

and let for every $a \in A$ and $m \in M$, $\sigma(ma) - \sigma(m)a = d_0(x)$. Then,

$$\sigma^1(\xi)(a, m) = -\mu(\xi_1(x))$$

where

$$\mu : F_0 \longrightarrow N$$

is the augmentation morphism of F_* . Then,

$$\sigma^1(\xi) \in \text{Der}_k(A, \text{Hom}_k(M, N))$$

and its class in $HH^1(A, \text{Hom}_k(M, N))$ represents ξ .

Recall the spectral sequence associated to a change of rings. If $\pi : A \longrightarrow B$ is a surjective homomorphism of commutative k -algebras, M a B -module and N an A -module, then $\text{Ext}_A^*(M, N)$ is the abutment of the spectral sequence given by,

$$E_2^{p,q} = \text{Ext}_B^p(M, \text{Ext}_A^q(B, N)).$$

There is an exact sequence,

$$0 \longrightarrow E_2^{1,0} \longrightarrow \text{Ext}_A^1(M, N) \longrightarrow E_2^{0,1} \longrightarrow E_2^{2,0}$$

Which, for a B -module N , considered as an A -module, implies the exactness of

$$\begin{aligned} 0 &\longrightarrow \text{Ext}_B^1(M, N) \longrightarrow \text{Ext}_A^1(M, N) \\ &\longrightarrow \text{Hom}_B(M, \text{Hom}_B(I/I^2, N)) \longrightarrow \text{Ext}_B^2(M, N) \end{aligned}$$

where $I = \ker \pi$. The corresponding exact sequence,

$$\begin{aligned} 0 &\longrightarrow HH^1(B, \text{Hom}_k(M, N)) \longrightarrow HH^1(A, \text{Hom}_k(M, N)) \\ &\longrightarrow \text{Hom}_{A \otimes A^{op}}(B, \text{Hom}_k(M, N)) \end{aligned}$$

in the non commutative case is induced by the sequence ,

$$\begin{aligned} 0 &\longrightarrow \text{Der}_k(B, \text{Hom}_k(M, N)) \longrightarrow \text{Der}_k(A, \text{Hom}_k(M, N)) \\ &\longrightarrow \text{Hom}_{A \otimes A^{op}}(B, \text{Hom}_k(M, N)) \end{aligned}$$

Notice that in general we do not know that the last morphism is surjective. This, however, is true if $B = A/\text{rad}(A)$, where $\text{rad}(A)$ is the radical of A , and A is a finite dimensional, i.e. an artinian, k -algebra. In this case, B is semisimple and the

surjectivity above follows from the Wedderburn-Malcev theorem. Notice also that in the commutative case,

$$\operatorname{Hom}_{A \otimes A^{\text{op}}}(B, \operatorname{Hom}_k(M, N)) \simeq \operatorname{Hom}_B(I/I_2, \operatorname{Hom}_B(M, N))$$

as it must, since for $\phi \in \operatorname{Hom}_B(M, N)$, $a \in A$, and $b \in I$, $ab=ba$, and therefore

$$(a\phi)b = \phi(ab) = \phi(ba) = (\phi a)(b)$$

This implies that for $B = A/\mathfrak{p}$, $M = A/\mathfrak{p}$, $N = A/\mathfrak{q}$, where $\mathfrak{p} \subseteq \mathfrak{q}$ are (prime) ideals of A ,

$$\operatorname{Ext}_A^1(A/\mathfrak{p}, A/\mathfrak{q}) \simeq \operatorname{Hom}_A(\mathfrak{p}/\mathfrak{p}^2, A/\mathfrak{q})$$

and, in particular

$$\operatorname{Ext}_A^1(A/\mathfrak{q}, A/\mathfrak{q}) \simeq \operatorname{Hom}_A(\mathfrak{q}/\mathfrak{q}^2, A/\mathfrak{q}) = N_{\mathfrak{q}},$$

the normal bundle of $V(\mathfrak{q})$ in $\operatorname{Spec}(A)$. If $\mathfrak{q} \subset \mathfrak{p}$ and $\mathfrak{q} \neq \mathfrak{p}$ we find,

$$\operatorname{Ext}_A^1(A/\mathfrak{p}, A/\mathfrak{q}) \simeq \operatorname{Ext}_{A/\mathfrak{q}}^1(A/\mathfrak{p}, A/\mathfrak{q}).$$

In [12], chapter 1., we considered the cohomology of a category $\underline{\mathcal{C}}$ with values in a bifunctor, i.e. in a functor defined on the category $\operatorname{mor} \underline{\mathcal{C}}$. It is easy to see that this is an immediate generalization of the projective limit functor and its derivatives, or, if one likes it better, the obvious generalization of the Hochschild cohomology of a ring. In fact, for every small category $\underline{\mathcal{C}}$ and for every bifunctor,

$$G : \underline{\mathcal{C}} \times \underline{\mathcal{C}} \longrightarrow Ab$$

contravariant in the first variable, and covariant in the second, one obtains a covariant functor,

$$G : \operatorname{mor} \underline{\mathcal{C}} \longrightarrow Ab.$$

Consider now the complex,

$$D^*(\underline{\mathcal{C}}, G)$$

where,

$$D^p(\underline{\mathcal{C}}, G) = \prod_{c_0 \rightarrow c_1 \cdots \rightarrow c_p} G(c_0, c_p)$$

where the indices are strings of morphisms $\psi_i : c_i \rightarrow c_{i+1}$ in $\underline{\mathcal{C}}$, and the differential,

$$d^p : D^p(\underline{\mathcal{C}}, G) \longrightarrow D^{p+1}(\underline{\mathcal{C}}, G)$$

is defined as usual,

$$\begin{aligned} (d^p \xi)(\psi_1, \dots, \psi_i, \psi_{i+1}, \dots, \psi_{p+1}) &= \psi_1 \xi(\psi_2, \dots, \psi_{p+1}) \\ &+ \sum_{i=1}^p (-1)^i \xi(\psi_1, \dots, \psi_i \circ \psi_{i+1}, \dots, \psi_{p+1}) + (-1)^{p+1} \xi(\psi_1, \dots, \psi_p) \psi_{p+1}. \end{aligned}$$

As shown in [12], the cohomology of this complex is the higher derivatives of the projective limit functor $\varprojlim_{\operatorname{mor} \underline{\mathcal{C}}}^{(*)}$ applied to the covariant functor

$$G : \operatorname{mor} \underline{\mathcal{C}} \longrightarrow Ab.$$

This is the "Hochschild" cohomology of the category $\underline{\mathcal{C}}$, denoted

$$H^*(\underline{\mathcal{C}}, G) := H^*(D^*(\underline{\mathcal{C}}, G)).$$

Example 1.1. Let \underline{c} be a multiplicative subset of a ring R , considered as a category with one object, and let $\tilde{R} : \underline{c} \times \underline{c} \longrightarrow Ab$ be the functor, where for $\psi \in \underline{c}$, $\tilde{R}(\psi, id) = \psi^*$ is left multiplication by ψ , and where $\tilde{R}(id, \psi) = \psi_*$ is right multiplication by ψ , then

$$H^0(\underline{c}, \tilde{R}) = \{\phi \in R \mid \phi\psi = \psi\phi \text{ for all } \psi \in \underline{c}\},$$

i.e. the commutant of \underline{c} in R .

Given a k -algebra A , and consider a subcategory \underline{c} of the category of right A -modules. Let $\pi : \underline{c} \rightarrow k - mod$ be the forgetful functor, and consider the bifunctor,

$$Hom_\pi : \underline{c} \times \underline{c} \longrightarrow k - mod$$

defined by

$$Hom_\pi(V_i, V_j) = Hom_k(V_i, V_j)$$

Put,

$$O_0(\underline{c}, \pi) := H^0(\underline{c}, Hom_\pi).$$

It is clear that $O_0(\underline{c}, \pi)$ is a k -algebra, and that there is a canonical homomorphism of k -algebras,

$$\eta_0(\underline{c}) : A \longrightarrow O_0(\underline{c}, \pi),$$

see section 2.

Example 1.2. Let A be a commutative k -algebra of finite type, k algebraically closed, and let $Spec(A)$ be the subcategory of A -mod consisting of the modules A/\mathfrak{p} , where \mathfrak{p} runs through $Spec(A)$, the morphisms being only the obvious ones. It is easy to see that the homomorphism

$$\eta(Spec(A), \pi) : A \longrightarrow O_0(Spec(A), \pi)$$

identifies $A/rad(A)$ with $O_0(Spec(A), \pi)$. If A is reduced we even find an isomorphism,

$$\eta(Simp^*(A), \pi) : A \simeq O_0(Spec(A), \pi).$$

Here $Simp^*(A)$ is the subcategory of A -mod where the objects are A and the simple A -modules, A/\mathfrak{m} , and the morphisms are the obvious quotient morphisms $A \rightarrow A/\mathfrak{m}$.

If, however, A is a local k -algebra, essentially of finite type, say, then this is no longer true in general.

To remedy this situation we shall in the next paragraph introduce, and study a generalization $O(\underline{c}, \pi)$ of $O_0(\underline{c}, \pi)$ defined in terms of the non-commutative deformation theory introduced in [16].

The category of A-G-modules. Let A be any k -algebra and let $g : A \rightarrow A$ be an automorphism. Given an A -module M_i , $i=1,2$ consider an automorphism of k -modules $\nabla_g^i : M_i \rightarrow M_i$, such that for $m_i \in M_i$ and $a \in A$ we have,

$$\nabla_g^i(m_i a) = \nabla_g^i(m_i) g(a) \text{ for } i=1,2$$

i.e. such that ∇_g^i is g -linear. Then there is an automorphism,

$$\theta_g^p := \theta_g^p(\nabla^1, \nabla^2) : Ext_A^p(M_1, M_2) \longrightarrow Ext_A^p(M_1, M_2)$$

induced via the isomorphism,

$$Ext_A^p(M_1, M_2) \simeq HH^p(A, Hom_k(M_1, M_2))$$

by the g^{-1} -linear automorphism of bi-modules,

$$\zeta_g : Hom_k(M_1, M_2) \longrightarrow Hom_k(M_1, M_2)$$

defined by,

$$\psi \longmapsto \nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2.$$

Notice that we compose morphisms in the natural order. For $a \in A$ we compute,

$$\zeta_g(g(a)\psi) = \nabla_g^1 \circ g(a)\psi \circ \nabla_{g^{-1}}^2 = a(\nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2) = a\zeta_g(\psi)$$

$$\zeta_g(\psi g(a)) = \nabla_g^1 \circ \psi g(a) \circ \nabla_{g^{-1}}^2 = (\nabla_g^1 \circ \psi \circ \nabla_{g^{-1}}^2)a = \zeta_g(\psi)a.$$

This implies that there is an automorphism of Hochschild cohomology,

$$\zeta_g^p : HH^p(A, Hom_k(M_1, M_2)) \longrightarrow HH^p(A, Hom_k(M_1, M_2))$$

defined on cochain form by,

$$\xi^p \longmapsto \{(a_1, a_2, \dots, a_p) \mapsto \nabla_g^1 \circ \xi^p(g(a_1), \dots, g(a_p)) \circ \nabla_{g^{-1}}^2\}.$$

In particular the automorphism,

$$\zeta_g^1 : Ext_A^1(M_1, M_2) \longrightarrow Ext_A^1(M_1, M_2)$$

is induced by the map

$$\zeta_g^1 : Der_k(A, Hom_k(M_1, M_2)) \longrightarrow Der_k(A, Hom_k(M_1, M_2))$$

defined by

$$\zeta_g^1(\delta)(a) = \nabla_g^1 \circ \delta(g(a)) \circ \nabla_{g^{-1}}^2.$$

When $\mathfrak{p} \subseteq A$ is a g -invariant ideal of A contained in the annihilator of M_2 , we know that the restriction of the derivations of $Der_k(A, Hom_k(M_1, M_2))$ to \mathfrak{p} induces an isomorphism,

$$Hom_A(\mathfrak{p}/\mathfrak{p}^2, Hom_A(A/\mathfrak{p}, M_2)) \simeq Ext_A^1(A/\mathfrak{p}, M_2)$$

such that the automorphism ζ_g^1 takes the form,

$$\zeta_g^1(\psi)(x) = \nabla_{g^{-1}}^2(\psi(gx)) \text{ for } x \in \mathfrak{p}/\mathfrak{p}^2.$$

Suppose $\xi \in Ext_A^1(M_1, M_2)$ is represented by the exact sequence of A-modules,

$$(*) \quad 0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

Since the g-linear automorphisms $\nabla_g^i : M_i \rightarrow M_i$ correspond to an A-linear isomorphism,

$$\nabla_g^i : M_i \rightarrow M_i \otimes_{g^{-1}} A$$

we deduce from (*) the exact sequence of A-modules,

$$(**) \quad 0 \longrightarrow M_2 \otimes_{g^{-1}} A \longrightarrow E \otimes_{g^{-1}} A \longrightarrow M_1 \otimes_{g^{-1}} A \longrightarrow 0$$

which represents the element $\zeta_g^1(\xi) \in Ext_A^1(M_1, M_2)$. The ζ_g^1 -invariant elements ξ of $Ext_A^1(M_1, M_2)$ therefore corresponds to the extensions (*) for which there exists an isomorphism

$$(***) \quad \nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$$

compatible with the ∇_g^i , for $i=1,2$. Another way of viewing this is to look at $\zeta_g^1(\xi) - \xi$ as an obstruction for the existence of such an isomorphism (**).

Given one $\nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$ compatible with the ∇_g^i , another ∇_g' will differ from the first one by the composition Γ_g of the homomorphism $E \longrightarrow M_1$ and some A-linear map $\alpha : M_1 \rightarrow M_2 \otimes_{g^{-1}} A$, and any such Γ_g added to (**), will again be compatible with the ∇_g^i , for $i=1,2$. In the category of (A-g)-modules, we therefore find,

$$Ext_{A-g}^1(M_1, M_2) \simeq Ext_A^1(M_1, M_2)^{\zeta_g} \oplus Hom_A(M_1, M_2 \otimes_{g^{-1}} A) / \sim$$

The equivalence \sim identifies (E', ∇_g') and (E'', ∇_g'') if there exists an isomorphism of extensions $\zeta : E \simeq E''$ compatible with the ∇_g 's. Since

$$\nabla_g^2 : Hom_A(M_1, M_2) \simeq Hom_A(M_1, M_2 \otimes_{g^{-1}} A)$$

the equivalence relation \sim is trivial.

Now, suppose G is a group acting on the k-algebra A, i.e. suppose there exists a homomorphism of groups,

$$\rho : G \longrightarrow Aut_k(A).$$

Consider A-modules M_i , $i=1,2$, with G-actions compatible with ρ , i.e. homomorphisms

$$\nabla^i : G \longrightarrow Aut_k(M_i)$$

such that for $g \in G$, $m_i \in M_i$, and $a \in A$,

$$\nabla_g^i(m_i a) = \nabla_g^i(m_i) g(a) \text{ for } i=1,2$$

where we denote by $g(a)$ the action of $\rho(g)$ on $a \in A$.

Given an invariant $\xi \in \text{Ext}_A^1(M_1, M_2)$ under the action of the group G , as explained above, there exists for every $g \in G$ an isomorphism

$$\nabla_g : E \longrightarrow E \otimes_{g^{-1}} A$$

Since

$$(E \otimes_{g_1^{-1}} A) \otimes_{g_2^{-1}} A = E \otimes_{(g_1 g_2)^{-1}} A$$

we find an obstruction for the existence of a homomorphism of groups,

$$\nabla : G \longrightarrow \text{Aut}_k(E)$$

compatible with the ∇^i 's which is a 2-cocycle of G with values in the G -bimodule $\text{Hom}_A(M_1, M_2)$,

$$(g_1, g_2) \longmapsto (\nabla_{g_1} \circ \nabla_{g_2} - \nabla_{g_1 g_2}).$$

When the corresponding 2-class,

$$\sigma_\xi \in H^2(G, \text{Hom}_A(M_1, M_2))$$

vanish, there exists a ∇ and the set of such will be a torsor under

$$H^1(G, \text{Hom}_A(M_1, M_2))$$

.

Proposition 1.3. Suppose $H^i(G, \text{Hom}_A(M_1, M_2)) = 0$ for $i=1,2$, then,

$$\text{Ext}_{A-G}^1(M_1, M_2) \simeq \text{Ext}_A^1(M_1, M_2)^G.$$

Notice that a 1-coboundary of the form

$$g \longmapsto (g\alpha - \alpha)$$

corresponds to an automorphism $\theta_\alpha : E \longrightarrow E$ inducing an automorphism of (E, ∇_g) .

The category of A - \mathfrak{g} -modules. Suppose

$$\rho : \mathfrak{g} \longrightarrow \text{Der}_k(A)$$

is a k -Lie homomorphism, e.g. a Lie-Cartan pair. We shall treat this as the tangent map of a Lie-group action ρ studied in the previous section. Let M_i , $i=1,2$ be A -modules with \mathfrak{g} -integrabel connections

$$\nabla^i : \mathfrak{g} \longrightarrow \text{End}_k(M_i),$$

and consider for every $\delta \in \mathfrak{g}$ and every $\psi \in \text{Hom}_k(M_1, M_2)$ the map

$$\delta \longmapsto \nabla_\delta^1 \psi - \psi \nabla_\delta^2$$

This defines a Lie algebra homomorphism,

$$\rho : \mathfrak{g} \longrightarrow \text{End}_k(\text{Hom}_k(M_1, M_2))$$

such that, if ρ is a Lie-Cartan pair, $\rho(\delta a) = a\rho(\delta) - \rho(\delta)a$.

Let $D \in \text{Der}_k(A, \text{Hom}_k(M_1, M_2))$, then the map

$$a \longmapsto \nabla_\delta(D)(a) := D(\delta(a)) + \nabla_\delta^1 D(a) - D(a)\nabla_\delta^2$$

is a derivation, and we obtain a connection

$$\nabla : \mathfrak{g} \longrightarrow \text{End}_k(\text{Ext}_A^1(M_1, M_2))$$

As above, every $\xi \in \text{Ext}_A^1(M_1, M_2)^\mathfrak{g}$ is associated to an obstruction,

$$\sigma(\xi) \in H^2(\mathfrak{g}, \text{Hom}_k(M_1, M_2))$$

which vanish if and only if there exists an integrabel connection on the middle term E of the exact sequence representing ξ ,

$$0 \longrightarrow M_2 \longrightarrow E \longrightarrow M_1 \longrightarrow 0$$

compatible with the connections ∇^i on M_i . The set of isomorphism classes of such (ξ, ∇) is then a torsor under

$$H^1(\mathfrak{g}, \text{Hom}_A(M_1, M_2))$$

Proposition 1.4. *Suppose*

$$H^i(\mathfrak{g}, \text{Hom}_A(M_1, M_2)) = 0 \text{ for } i=1,2$$

then,

$$\text{Ext}_{A-\mathfrak{g}}^1(M_1, M_2) = \text{Ext}_A^1(M_1, M_2)^\mathfrak{g}$$

2.Non-commutative schemes.

Trivializations and observables.. Let \underline{C} be any abelian category with Massey products. The last proviso is satisfied if \underline{C} has enough projectives, but there are other cases where Massey products exist even though projectives are scarce. See [13] and [30] for an exposition of the Massey product structure in the category of all O_X -modules for X a scheme defined on some field k . Let $\underline{c} \subseteq \underline{C}$ be a diagram. Assume there exists an exact and faithful functor

$$\pi : \underline{c} \longrightarrow k\text{-mod.}$$

Definition 2.1. Any such functor π will be called a *trivialization* of \underline{c} .

Example 2.2. The obvious example of this set up is the following: Let A be any k -algebra, k a field, put $\underline{C} = A - \text{mod}$ and let

$$\pi : A - \text{mod} \longrightarrow k - \text{mod}.$$

be the forgetful functor. Then π will be a trivialization for any diagram

$$\underline{c} \subseteq \underline{C} = A - \text{mod}.$$

Unless we specifically mention another choice of trivialization, this is the one we shall use in the sequel.

Fix the trivialization π of $\underline{c} \subseteq \underline{C}$, and consider the k -algebra

$$O_0(\underline{c}, \pi) := H^0(\underline{c}, \text{Hom}_\pi)$$

defined in (1). Recall that

$$\text{Hom}_\pi : \text{mor } \underline{c} \longrightarrow k - \text{mod}.$$

is the functor defined by

$$\text{Hom}_\pi(\psi) = \text{Hom}_k(\pi(c_1), \pi(c_2))$$

for $\psi : c_1 \rightarrow c_2$ in \underline{c} .

Definition 2.3. $O_0 := O_0(\underline{c}, \pi)$ is the k -algebra of *immediate observables*

It is clear that O_0 acts on each object $\pi(c) \in k - \text{mod}$, $c \in \text{ob } \underline{c}$, in the sense that there is a canonical k -algebra homomorphism

$$O_0 \longrightarrow \text{End}_k(\pi(c))$$

such that the image diagram

$$\text{im } \pi|_{\underline{c}} \subseteq k - \text{mod}$$

becomes a diagram of O_0 -representations.

In the example above, we obtain for every diagram $\underline{c} \subseteq A - \text{mod}$, a k -algebra $O_0(\underline{c}, \pi)$ acting on every A -module in \underline{c} such that \underline{c} becomes a diagram of $O_0(\underline{c}, \pi)$ -modules. Moreover there is a canonical homomorphism of k -algebras

$$\eta_0 : A \longrightarrow O_0(c, \pi)$$

which is, in an obvious sense, a universal "extension" of the algebra A , with respect to the diagram \underline{c} . Since we have,

$$\underline{c} \subseteq O_0 - \text{mod}$$

and since the trivialization π induces a trivialization,

$$\pi_0 : O_0 - \text{mod} \longrightarrow k - \text{mod}$$

we may repeat the construction of trivial observables. We obtain,

$$O_0(\underline{c}, \pi_0) = O_0(\underline{c}, \pi) = O_0$$

This implies that the operation of constructing trivial observables, is a closure operation.

Example 2.4. Consider any reduced commutative k -algebra A of finite type. Recall from (1.2) that if $\underline{c} = \text{Spec}(A)$, or if $\underline{c} = \text{Simp}^*(A)$ then

$$\eta_0 : A \longrightarrow O_0(\underline{c}, \pi)$$

is an isomorphism, provided k is algebraically closed. Denote by $\text{Ind}(A)$ the full subcategory of $A\text{-mod}$ defined by the indecomposable modules and let $\text{Prim}(A)$ denote the subdiagram of $\text{Ind}(A)$ composed by the A -modules of the form A/\mathfrak{q} , where \mathfrak{q} is a primary ideal, and where the morphisms are the obvious ones. It is easy to see that the canonical homomorphism

$$\eta_0 : A \longrightarrow O_0(\text{Prim}(A), \pi)$$

is an isomorphism when A is reduced. However, this is not true in general, just look at the case $A = k[\epsilon]$, where

$$O_0(\text{Prim}(A), \pi) \simeq \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}.$$

Notice that there is a generalized Zariski topology on $\text{Prim}(A)$, due to Jacobson, defined as follows. Let $a \in A$ and consider the full subdiagram $D(a)$ of $\text{Prim}(A)$ defined by the objects V for which a is not a zerodivisor. Obviously $D(a) \cap D(b) = D(ab)$ and $D(a)$ is simply the localization of $\text{Prim}(A)$ at $\{a\}$. There are canonical isomorphisms

$$O_0(D(a), \pi) \simeq A_{(a)} = O_S(D(a)|_{\text{Spec}(A)})$$

where S is the affine scheme $\text{Spec}(A)$, and where O_S is the structure sheaf. This shows that there exists a ringed space $(\text{Prim}(A), O_P)$, and a continuous map

$$S = \text{Spec}(A) \longrightarrow \text{Prim}(A) = P$$

inducing isomorphisms of the structure sheaves

$$O_S \simeq O_P.$$

The problem with $\text{Prim}(A)$ is that it is too big, that the topology is too coarse, and that it has some unsatisfactory functorial properties. On the other hand, $\text{Spec}(A)$ and $\text{Simp}^*(A)$ seem to be too small since the trivial observables for $\text{Spec}(A)$ kills the nilpotents of A , even for finite type k -algebras, and $\eta_0 : A \rightarrow O_0(\text{Simp}^*(A), \pi)$ is far from an isomorphism when A is local.

These problems stem from the trivial nature of the trivial observables. In the construction of O_0 , we use only the trivial categorical structure of $A\text{-mod}$, restricted to \underline{c} . To get to the goal, we have to take into account the infinitesimal structure of the category $A\text{-mod}$, i.e. the abelian structure of $A\text{-mod}$, and, in particular, the family of multiple extensions of the objects of \underline{c} .

The goal is to construct, for every diagram \underline{c} , an extension of $O_0(\underline{c}, \pi)$, which we shall denote $\mathcal{O}_\pi(\underline{c})$, and canonical homomorphisms

$$A \xrightarrow{\eta} \mathcal{O}_\pi(\underline{c}) \xrightarrow{\rho} O_0(\underline{c}, \pi)$$

extending η_0 . We shall show that \mathcal{O}_π , has good functorial properties, mimicking the notion of *structure sheaf* in commutative algebra, and providing us with a generalized, non-commutative, algebraic geometry. We shall be guided by the principles of the main Introduction.

So consider a diagram \underline{c} in $\underline{C} = A - \text{mod}$, together with the forgetfull functor π . Assume first that \underline{c} is finite. Let $|\underline{c}| = \{V_i\}_{i=1}^r$, be the family of objects, and construct the non-commutative formal moduli $H(|\underline{c}|) = (H_{i,j})$ as in [16]. Let $\tilde{V} = (H_{i,j} \otimes V_j)$ be the versal family and consider the k-algebra

$$O(|\underline{c}|, \pi) := \text{End}_H(\tilde{V}) = (H_{i,j} \otimes \text{Hom}_k(V_i, V_j))$$

and the k-algebra homomorphism,

$$\eta(|\underline{c}|) : A \longrightarrow O(|\underline{c}|, \pi)$$

defined by the action of A on \tilde{V} , which, by construction, commutes with the action of $H(|\underline{c}|)$.

Recall that the non commutative formal moduli is unique up to isomorphisms, and that having fixed a versal family, as a deformation, the action of A on \tilde{V} is unique up to isomorphisms. This means that for any other homomorphism

$$\eta(|\underline{c}|)' : A \longrightarrow O(|\underline{c}|, \pi)$$

defining the same deformation, there exists an automorphism

$$\omega \in (H_{i,j} \otimes \text{Hom}_k(V_i, V_j))$$

such that

$$\eta(|\underline{c}|)' = \omega \eta(|\underline{c}|) \omega^{-1}.$$

Notice that ω , as an element of $O(|\underline{c}|, \pi)$, is a unit.

Recall also that, for an artinian algebra A, and for the family \mathcal{V} of all the simple A-modules $\eta(\mathcal{V})$ is an isomorphism, (In the preprint [18] $O(|c|, \pi)$ was denoted $A(\mathcal{V})$).

Notice that, by definition of the terms, there is a unique morphism of k-algebras,

$$\rho_0 : O(|\underline{c}|, \pi) \longrightarrow O_0(|\underline{c}|, \pi)$$

which, together with η and η_0 form a commutative diagram. Therefore $|\underline{c}|$ is, in an obvious sense, a family of $O(|\underline{c}|, \pi)$ -modules. Notice also that if $\underline{c}_1 \subseteq \underline{c}_2$, there exist a canonical surjective homomorphism

$$H(|\underline{c}_2|) \longrightarrow H(|\underline{c}_1|).$$

induced by the functors,

$$\underline{a}_{r_1} \rightarrow \underline{a}_{r_2} \rightarrow \underline{a}_{r_1}$$

where r_i , $i = 1, 2$, is the number of objects in \underline{c}_i . At the tangent level this morphism corresponds to the inclusion,

$$(\text{Ext}_A^1(V_i, V_j))_{i,j=1,\dots,r_1} \subseteq (\text{Ext}_A^1(V_i, V_j))_{i,j=1,\dots,r_2}.$$

Beware, in general, this k-algebra homomorphism admits no sections!

If \underline{c} is infinite we put

$$O(|\underline{c}|, \pi) = \varprojlim_{\underline{c}_0 \subseteq \underline{c}} O(|\underline{c}_0|, \pi)$$

where \underline{c}_0 runs through all finite subdiagrams of \underline{c} . The k-algebra we are heading for is now a subquotient of $O(|\underline{c}|, \pi)$, singled out by the π -incidences of our geometry, i.e. by the morphisms

$$\phi_{i,j} : V_i \rightarrow V_j$$

of our diagram.

Let $\Gamma(\underline{c})$ be the quiver corresponding to \underline{c} , i.e. with set of nodes equal to the set of objects of \underline{c} , and with arrows corresponding to the morphisms $\phi_{i,j}$ of \underline{c} . Notice that \underline{c} is a diagram of \underline{C} not a subcategory, therefore we do not require that the identities of the objects be morphisms of \underline{c} . Corresponding to $\Gamma(\underline{c})$ there is the universal k-algebra $k[\Gamma(\underline{c})]$, and corresponding to a component $\Gamma_p(\underline{c})$ of $\Gamma(\underline{c})$, there is a subdiagram $\underline{c}(p)$ of \underline{c} .

Assume first that \underline{c} is finite, and assume that $\Gamma(\underline{c})$ (or \underline{c}) is connected. Put $r =$ number of elements in \underline{c} . Consider the obvious representation of $k[\Gamma(\underline{c})]$ on $V := \bigoplus_{i=1}^r V_i$. The image of $k[\Gamma(\underline{c})]$ in $\text{End}_A(\bigoplus_{i=1}^r V_i)$ is the k-algebra $\text{end}(\underline{c})$ generated by the morphisms of \underline{c} . Now $V = \bigoplus_{i=1}^r V_i$ is an $k[\Gamma(\underline{c})] \otimes A$ -module, and as such an A -module, as well as a $k[\Gamma(\underline{c})]$ -module. We may consider the ordinary (non-commutative) deformation functors of this module. Let the formal moduli, the prorepresenting hulls of these functors be, $H(\underline{c})$, $H(V)$ respectively $H^\Gamma(V)$. There is a (non unique) natural morphism,

$$h(1) : H^\Gamma(V) \rightarrow H(\underline{c}).$$

Let $H(\underline{c})_0$ be the $k[\Gamma] \otimes_k A$ -modular *substratum* of $H(\underline{c})$, see e.g. [15], and consider the obvious composition

$$h(2) : H^\Gamma(V) \rightarrow H(\underline{c})_0.$$

Recall that the modular, or prorepresentable, substratum $H(\underline{c})_0$ of $H(\underline{c})$ is the candidate for the completed local ring of the (usually non-existing) moduli scheme for $A \otimes k[\Gamma]$ -modules at the point corresponding to V . It is the unique *substratum*, i.e. quotient, of the formal moduli $H(\underline{c})$ such that the composition

$$\text{Mor}(H(\underline{c})_0, -) \rightarrow \text{Mor}(H(\underline{c}), -) \rightarrow \text{Def}_V$$

is injectiv. There is a universal deformation of V to $H(\underline{c})_0$, i.e. an action of $k[\Gamma] \otimes_k A$ on $H(\underline{c})_0 \otimes V$, uniquely inducing all other modular deformations.

Another characterization of the modular stratum is that it is the unique maximal quotient of the formal moduli on which the Kodaira-Spencer morphism

$$\text{Der}_k(H(\underline{c})_0) \rightarrow \text{Ext}_{A \otimes k[\Gamma]}^1(H(\underline{c})_0 \otimes V, H(\underline{c})_0 \otimes V)$$

is injectiv, see ([15], Chapter 2.).

Finally, let $H_p(\underline{c})$ be the cokernel of the morphism $h(p)$, for $p = 0, 1$. This is an (up to isomorphisms) unique common quotient of $H(V)$ and $H(\underline{c})$. Moreover,

since the $k[\Gamma]$ -action on $H_0(\underline{\mathcal{C}}) \otimes V$ induced by the universal action of $k[\Gamma] \otimes_k A$ on $H(\underline{\mathcal{C}})_0 \otimes V$, by construction, corresponds to the trivial morphism,

$$H^\Gamma(V) \rightarrow k \rightarrow H_0(\underline{\mathcal{C}}),$$

there exists an isomorphism class of universal actions of $k[\Gamma] \otimes_k A$ on $H_0(\underline{\mathcal{C}}) \otimes V$, containing at least one action

$$\eta_0(\underline{\mathcal{C}}) : k[\Gamma] \otimes_k A \rightarrow \text{End}_{H(\underline{\mathcal{C}})_0}(H(\underline{\mathcal{C}})_0 \otimes V) = (H(\underline{\mathcal{C}})_0 \otimes \text{Hom}_k(V_i, V_j))$$

for which the induced Γ -action is the trivial one. Denote by $\eta_0(V)$ the A -action on $H_0(\underline{\mathcal{C}}) \otimes V$, induced by $\eta_0(\underline{\mathcal{C}})$.

Now, given any deformation ξ_S of V to some k -algebra $S \in \underline{\mathcal{A}}_1$, denote by \underline{m} the maximal ideal of S . Let \tilde{S} be the r -pointed matrix k -algebra $(\tilde{S}_{i,j})$ where $\tilde{S}_{i,i} = S$, on the diagonal, and $\tilde{S}_{i,j} = \underline{m}$ at the other places, i.e. for $i \neq j$. Clearly \tilde{S} is in $\underline{\mathcal{A}}_r$, and the i -th. line of the matrix $(\tilde{S}_{i,j} \otimes V_j)$ is

$$(\underline{m} \otimes V_1) \oplus \cdots \oplus (S \otimes V_i) \oplus \cdots \oplus (\underline{m} \otimes V_r) \subseteq H(V) \otimes V.$$

Let $v_i \in V_i$, and $a \in A$. The component of $(1 \otimes v_i)a$ in $S \otimes V_j$ for $i \neq j$ sits in $\underline{m} \otimes V_j$. This shows that A acts on each line of the matrix $(\tilde{S}_{i,j} \otimes V_j)$, commuting with the left action of $(\tilde{S}_{i,j})$, implying that $(\tilde{S}_{i,j} \otimes V_j)$ is, in a natural way, a non-commutative deformation of the family of right A -modules $|\underline{\mathcal{C}}|$, to \tilde{S} . Therefore there is a morphism,

$$\iota_S : H(|\underline{\mathcal{C}}|) \longrightarrow \tilde{S}$$

compatible with the specified deformations of right A -modules. This induces a morphism of k -algebras,

$$(H_{i,j} \otimes \text{Hom}_k(V_i, V_j)) \longrightarrow (\tilde{S}_{i,j} \otimes \text{Hom}_k(V_i, V_j)).$$

Since the right hand side k -algebra is a subalgebra of

$$\text{End}_S(S \otimes (\oplus_{i=1}^r V_i)) = (S \otimes \text{Hom}_k(V_i, V_j)),$$

we obtain a homomorphism of k -algebras,

$$\kappa_S : (H_{i,j} \otimes \text{Hom}_k(V_i, V_j)) \longrightarrow (S \otimes \text{Hom}_k(V_i, V_j)).$$

such that the action $\eta(|\underline{\mathcal{C}}|)$ is mapped to the A -action on $S \otimes (\oplus_{i=1}^r V_i)$ defining the deformation ξ_S . In particular, for the versal deformation of V to $H(V)$, and for the versal A -action on $H(V) \otimes (\oplus_{i=1}^r V_i)$, there is a homomorphism of k -algebras,

$$\kappa_{H(V)} : (H_{i,j} \otimes \text{Hom}_k(V_i, V_j)) \longrightarrow (H(V) \otimes \text{Hom}_k(V_i, V_j))$$

compatible with the actions. By construction of the terms involved, it is clear that $\kappa_{H(V)}$ is injective, and that $H(V)$ is generated by the images of the components $\iota_{i,j} : H_{i,j} \rightarrow H(V)$ of $\iota_{H(V)}$. Therefore we have the adjunction relation,

$$\text{Mor}_{\underline{\mathcal{A}}_r}(H(|\underline{\mathcal{C}}|), \tilde{S}) \simeq \text{Mor}_{\underline{\mathcal{A}}_1}(H(V), S).$$

Use this for the versal $k[\Gamma] \otimes A$ -action on $H_p(\underline{\mathcal{C}}) \otimes V$, and the corresponding A -action $\eta_p(V)$. There exists an isomorphism

$$\omega \in \text{Aut}_{H_p(\underline{\mathcal{C}})}(H_p(\underline{\mathcal{C}}) \otimes (\oplus_{i=1}^r V_i)) \subseteq (H_p(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)),$$

such that $\kappa_{H_p(\underline{\mathcal{C}})}$, composed with the inner automorphism $\omega - \omega^{-1}$ of $(H_p(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j))$, is consistent with the actions $\eta(|\underline{\mathcal{C}}|)$ and $\eta_p(V)$. Call the composition κ_p . Notice that, since the action $\eta_p(\underline{\mathcal{C}})$, in general, is not unique, and since there is a choice of ω , κ_p is far from unique.

Now, depending on the choice of $p = 1, 2$, we define,

Definition 2.5. The k -algebra of *preobservables* $O(\underline{\mathcal{C}}, \pi)$ of the finite diagram $\underline{\mathcal{C}}$, is the subalgebra of $(H_{i,j}(|\underline{\mathcal{C}}|) \otimes \text{Hom}_\pi(V_i, V_j))$ commuting, via κ_S , with the induced actions of $k[\Gamma(\underline{\mathcal{C}})]$ in the $k[\Gamma] \otimes_k A$ -module $(S \otimes \text{Hom}_k(V_i, V_j))$ for all quotients S of $H_p(\underline{\mathcal{C}})$.

Clearly, the morphism,

$$(5) \quad \eta(|\underline{\mathcal{C}}|) : A \longrightarrow (H_{i,j} \otimes \text{Hom}_\pi(V_i, V_j))$$

induces a natural homomorphism of k -algebras,

$$(6) \quad \eta(\underline{\mathcal{C}}, \pi) : A \longrightarrow O(\underline{\mathcal{C}}, \pi)$$

Moreover, this construction is, up to isomorphisms, independent upon the choices made. In fact, the versal non-commutative deformation on $(H_{i,j}(|\underline{\mathcal{C}}|) \otimes V_j)$ is unique up to isomorphisms. The choice of an A -action corresponds to the choice of a homomorphism $\eta(|\underline{\mathcal{C}}|)$. As we have seen, two such are related via an interior automorphism of $(H_{i,j}(|\underline{\mathcal{C}}|) \otimes \text{Hom}_\pi(V_i, V_j))$.

Since the definition above is equivalent to the following,

Definition 2.5, bis. The k -algebra of *preobservables* $O(\underline{\mathcal{C}}, \pi)$ of the finite diagram $\underline{\mathcal{C}}$, is the subalgebra of

$$(H_{i,j}(|\underline{\mathcal{C}}|) \otimes \text{Hom}_\pi(V_i, V_j))$$

commuting, via the morphism,

$$\kappa_S : (H_{i,j}(|\underline{\mathcal{C}}|) \otimes \text{Hom}_\pi(V_i, V_j)) \rightarrow (S \otimes \text{Hom}_k(V_i, V_j)),$$

induced by any surjectiv k -algebra homomorphism

$$H_p(\underline{\mathcal{C}}) \longrightarrow S,$$

with all the liftings to S of all the A -module endomorphisms $\phi_{i,j}$ of V defined by the morphisms of the diagram $\underline{\mathcal{C}}$.

-it is clear that $O(\underline{\mathcal{C}}, \pi)$ and $\eta(\underline{\mathcal{C}}, \pi)$ are uniquely defined, up to isomorphisms.

Remark 2.6. (a): The definition of $H_p(\underline{\mathcal{C}})$, $p = 0, 1$, are just some possible choices of a functorial quotient of $H(\underline{\mathcal{C}})$. We might just have picked the residue field. And we might have considered all morphisms $H_p(\underline{\mathcal{C}}) \rightarrow S$, not only the surjectives. This

would have led to a functor in the k -algebra A . The above definition seems, however, to be the most natural, extending classical constructions, and leading to an, up to isomorphisms, unique k -algebra $O(\underline{c}, \pi)$. It is easy to see that the tangent space of $H(\underline{c})$ is,

$$Ext_{A \otimes_k k[\Gamma]}^1(V, V) \subseteq Ext_A^1(V, V)^\Gamma \oplus Ext_{k[\Gamma]}^1(V, V)^A$$

given in terms of a well known spectral sequence. From this follows that the tangent space of $H_1(\underline{c})$ is a sub k -vectorspace of

$$Ext_A^1(V, V)^\Gamma = (Ext_A^1(V_i, V_j))^{end(\underline{c})},$$

and that the tangent space of $H_0(\underline{c})$ is a sub k -vectorspace of

$$(Ext_A^1(V, V)^\Gamma)^{End_{A \otimes_k k[\Gamma]}(V)}.$$

Compare with [16].

(b): Let

$$(7) \quad \phi_{i,j} : V_i \longrightarrow V_j$$

be a morphism of \underline{c} . Denote by $\{\phi\}$ the subdiagram of \underline{C} defined by (7), and let $\{V_i\}$, resp. $\{V_j\}$ be the subdiagrams defined by each one of the modules. Put,

$$H(|\{\phi\}|) = (H_{p,q}), \quad p, q \in \{i, j\}.$$

Since the formal moduli of the A -module V_l , in the non-commutative sense, i.e. the hull $H(V_l)$ of Def_{V_l} is equal to $H(\{V_l\})$, it follows that there are canonical surjective homomorphisms

$$H_{l,l} \longrightarrow H(V_l), \quad l = i, j.$$

Now, the morphism $\phi_{i,j}$ induces maps

$$H_{i,j} \longrightarrow H_{l,l}, \quad l = i, j$$

These are, respectively, left and right linear on $H_{l,l}$, for $l=i, j$. Both morphisms are defined in terms of Massey products with $\phi_{i,j}$, see [13,16,17]. Moreover, it follows from the construction of [13], properly generalized to the non-commutative case, that the formal moduli for the morphism $\phi_{i,j}$, in the sense of [12] is

$$(8) \quad H(\phi_{i,j}) \text{ mapping onto } H(V_i) \otimes_{H_{i,j}} H(V_j)$$

In particular, there exists a universal lifting of $\phi_{i,j}$,

$$(9) \quad \tilde{\phi}_{i,j} : H(\phi_{i,j}) \otimes V_i \longrightarrow H(\phi_{i,j}) \otimes V_j$$

and morphisms,

$$(10) \quad \iota_l : H(V_l) \otimes End_\pi(V_l) \longrightarrow H(\phi_{i,j}) \otimes End_\pi(V_l), \quad l = i, j.$$

which induce morphisms,

$$(11) \quad \nu_l : H(V_l) \otimes \text{End}_\pi(V_l) \longrightarrow H(\phi_{i,j}) \otimes \text{Hom}_\pi(V_i, V_j), \quad l = i, j.$$

To compute the k -algebra of (pre)observables for the diagram $\{\phi\}$ we must consider the quiver of the diagram. It looks like,

$$\Gamma(\{\phi\}) : \circ \xrightarrow{\gamma} \circ,$$

Clearly,

$$|\{\phi\}| = \{V_i, V_j\}$$

and the formal moduli of this family of A -modules has the form,

$$H(|\{\phi\}|) = \begin{pmatrix} H_{i,i} & H_{i,j} \\ H_{j,i} & H_{j,j} \end{pmatrix}.$$

The tangent space of $H(|\{\phi\}|)$ looks like

$$(\text{rad}H/\text{rad}^2H)^* = \begin{pmatrix} \text{Ext}_A^1(V_i, V_i) & \text{Ext}_A^1(V_i, V_j) \\ \text{Ext}_A^1(V_j, V_i) & \text{Ext}_A^1(V_j, V_j) \end{pmatrix}$$

Consider now the sum of the modules $V = V_i \oplus V_j$ and the action of Γ on V . Obviously $k[\Gamma] \simeq k[\gamma]$, the k -algebra of polynomials in one variable, acts by sending γ to the endomorphism $\phi \in \text{Hom}_k(V_i, V_j) \subseteq \text{End}_k(V)$. Since the k -algebra $k[\gamma]$ is free, the d^2 of the spectral sequence referred to above vanishes, and we find that the tangent space of the formal moduli $H(\{\phi\})$ of V , considered as an $k[\Gamma] \otimes A$ -module is,

$$\begin{aligned} \text{Ext}_{A \otimes k[\Gamma]}^1(V, V) &= \{(\xi_{ii}, \xi_{jj}) \in \text{Ext}_A^1(V_i, V_i) \times \text{Ext}_A^1(V_j, V_j) \mid \phi_* \xi_{ii} = \phi^* \xi_{jj}\} \\ &\oplus \text{Ext}_A^1(V_i, V_j) \oplus \{\xi_{ji} \in \text{Ext}_A^1(V_j, V_i) \mid \phi_* \xi_{ji} = \phi^* \xi_{ji} = 0\} \\ &\oplus \{\text{Hom}_A(V_i, V_i) \oplus \text{Hom}_A(V_j, V_j)\} / \{\phi_*(\mu_{ij}) - \phi^*(\mu_{ij})\} \\ &\oplus \text{Hom}_A(V_i, V_j) / \{\text{im}\{\phi_*, \text{im}\phi^*\} \oplus \text{Hom}_A(V_j, V_i)\} \\ &= \text{Ext}_A^1(V, V)^\phi \oplus \text{Ext}_{k[\Gamma]}^1(V, V)^A \end{aligned}$$

Consequently the tangent space of $H_1(\{\phi\})$ is,

$$\text{Ext}_A^1(V, V)^\phi.$$

and the tangent space of $H_0(\{\phi\})$ is,

$$(\text{Ext}_A^1(V, V)^\phi)^{\text{End}_{A \otimes k[\Gamma]}(V)}.$$

Moreover the tangent space of $H(V_i) \otimes_{H_{i,j}} H(V_j)$ is

$$\{(\xi_{ii}, \xi_{jj}) \in \text{Ext}_A^1(V_i, V_i) \times \text{Ext}_A^1(V_j, V_j) \mid \phi_* \xi_{ii} = \phi^* \xi_{jj}\}.$$

The morphism

$$H(|\{\phi\}|) \longrightarrow \tilde{H}_p(\{\phi\}), \quad p = 0, 1,$$

is easily unraveled using (9, 10, and 11) above. The corresponding homomorphism of k -algebras, mapping the matrix k -algebra,

$$(12) \quad \begin{pmatrix} H_{i,i} \otimes \text{Hom}_k(V_i, V_i) & H_{i,j} \otimes \text{Hom}_k(V_i, V_j) \\ H_{j,i} \otimes \text{Hom}_k(V_j, V_i) & H_{j,j} \otimes \text{Hom}_k(V_j, V_j) \end{pmatrix}$$

into the k -algebra,

$$(13) \quad \begin{pmatrix} H_p(\mathcal{L}) \otimes \text{Hom}_k(V_i, V_i) & H_p(\mathcal{L}) \otimes \text{Hom}_k(V_i, V_j) \\ H_p(\mathcal{L}) \otimes \text{Hom}_k(V_j, V_i) & H_p(\mathcal{L}) \otimes \text{Hom}_k(V_j, V_j) \end{pmatrix}$$

is therefore reasonably easy to compute. The versal lifting $\Phi_{i,j}$ of $\phi_{i,j}$ to $H_p(\mathcal{L})$, i.e. the action of $\gamma \in \Gamma$ on $H_p(\mathcal{L}) \otimes V$, is an element of the last k -algebra (13), and the k -algebra of (pre)observables $O(\{\phi\}, \pi)$ is thus the subalgebra of those elements of (12) whose images in (13) commute with all $\Phi_{i,j}$ in (13). In particular the elements $\alpha \in O(\{\phi\}, \pi)$ satisfy the condition,

$$(D_{i,j}) \quad \Phi_{i,j} \circ \alpha_{j,j} = \alpha_{i,i} \circ \Phi_{i,j}$$

see [18].

(c): Consider the special case of (b), where A is a finite type k -algebra, and $\phi : A \rightarrow k(x)$ is the canonical homomorphism of A onto its closed point $k(x)$. The tangent space of $H(|\mathcal{L}|)$ is

$$(\text{rad}H/\text{rad}^2H)^* = \begin{pmatrix} 0 & 0 \\ \text{Ext}_A^1(k, A) & \text{Ext}_A^1(k, k) \end{pmatrix}$$

and the tangent space of $H_p(\mathcal{L})$ looks like

$$(\text{rad}H_p/\text{rad}^2H_p)^* = \begin{pmatrix} 0 & 0 \\ \text{Ext}_A^1(k, A)^\phi & \text{Ext}_A^1(k, k) \end{pmatrix}, \quad \text{for } p = 0, 1,$$

where $\text{Ext}_A^1(k, A)^\phi$ is the kernel of $\phi_* : \text{Ext}_A^1(k, A) \rightarrow \text{Ext}_A^1(k, k)$. The morphism

$$O(|\{\phi\}|, \pi) \rightarrow \begin{pmatrix} H_p(\mathcal{L}) \otimes_k \text{End}_k(A) & H_p(\mathcal{L}) \otimes_k \text{Hom}_k(A, k) \\ H_p(\mathcal{L}) \otimes_k \text{Hom}_k(k, A) & H_p(\mathcal{L}) \end{pmatrix}$$

maps an element

$$\alpha = \begin{pmatrix} \alpha_{1,1} & 0 \\ \alpha_{2,1} & \alpha_{2,2} \end{pmatrix} \in O(|\{\phi\}|, \pi)$$

to an element of the same form,

$$\tilde{\alpha} = \begin{pmatrix} \alpha_{1,1} & 0 \\ \tilde{\alpha}_{2,1} & \alpha_{2,2} \end{pmatrix} \in \begin{pmatrix} H_p(\mathcal{L}) \otimes_k \text{End}_k(A) & H_p(\mathcal{L}) \otimes_k \text{Hom}_k(A, k) \\ H_p(\mathcal{L}) \otimes_k \text{Hom}_k(k, A) & H_p(\mathcal{L}) \end{pmatrix}$$

Moreover a versal lifting of $\phi_{1,2}$ has the form,

$$\Phi = \begin{pmatrix} 0 & \Phi_{1,2} \\ \Phi_{2,1} & \Phi_{2,2} \end{pmatrix} \in \begin{pmatrix} H_p(\underline{\mathcal{C}}) \otimes_k \text{End}_k(A) & H_p(\underline{\mathcal{C}}) \otimes_k \text{Hom}_k(A, k) \\ H_p(\underline{\mathcal{C}}) \otimes_k \text{Hom}_k(k, A) & H_p(\underline{\mathcal{C}}) \end{pmatrix}$$

Suppose now that,

$$\Phi \tilde{\alpha} = \tilde{\alpha} \Phi$$

then, in particular,

$$\Phi_{1,2} \tilde{\alpha}_{2,1} = 0,$$

which implies that $\tilde{\alpha}_{2,1} = 0$, and then

$$\Phi_{1,2} \alpha_{2,2} = \alpha_{1,1} \Phi_{1,2}.$$

Since $\alpha_{2,2} =: \alpha_x \in H_{2,2} = \hat{A}_{\{x\}}$ is the obvious multiplication endomorphism, and since $\Phi_{1,2}$ reduces to the obvious completion map, $\rho_x : A \rightarrow H_{2,2} = \hat{A}_{\{x\}}$ we find that the conditions on α are,

$$(14) \quad \tilde{\alpha}_{2,1} = 0, \quad \alpha_{1,1} \rho_x = \rho_x \alpha_x$$

for some $\alpha_x \in \hat{A}_{\{x\}}$, together with the condition

$$(15) \quad \Phi_{2,1} \alpha_{1,1} = \alpha_x \Phi_{2,1}.$$

O is a closure operation. The most important property of the *O*-construction is a kind of functoriality (up to isomorphisms) and the closure property, given by the following result:

Theorem 2.7. Let $\psi : A \rightarrow B$ be a k -algebra homomorphism, and let $\underline{\mathcal{C}}$ be a finite diagram of B -modules. Consider the *O*-constructions, $O^A(\underline{\mathcal{C}}, \pi)$, resp. $O^B(\underline{\mathcal{C}}, \pi)$.

a. Assume the natural morphism,

$$\psi^* : H^A(\underline{\mathcal{C}}) \rightarrow H^B(\underline{\mathcal{C}})$$

induces a surjectiv homomorphism

$$\psi_p^* : H_p^A(\underline{\mathcal{C}}) \rightarrow H_p^B(\underline{\mathcal{C}}).$$

Then there exists an, up to isomorphisms, unique extension of ψ , i.e. a commutative diagram,

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ \eta_A \downarrow & & \downarrow \eta_B \\ O^A(\underline{\mathcal{C}}, \pi) & \xrightarrow{O(\psi)} & O^B(\underline{\mathcal{C}}, \pi) \end{array}$$

b. There is a natural isomorphism,

$$O(\eta_A) : O^A(\underline{\mathcal{C}}, \pi) \rightarrow O^{O^A}(\underline{\mathcal{C}}, \pi)$$

implying that the O -construction is a closure operation, for $p = 0, 1$.

Proof. Let the non commutative formal moduli of the family of B-modules $\{V_i\} = |\underline{\mathcal{C}}|$, considered as A- and B-modules be H^A resp. H^B . Since the versal family of B-modules $(H_{i,j}^B \otimes V_j)$ is also a family of A-modules, there is a morphism, $H^A \rightarrow H^B$ inducing the morphism of families of A-modules, $(H_{i,j}^A \otimes V_j) \rightarrow (H_{i,j}^B \otimes V_j)$ consistent with the induced A-module structure on the latter.

In the same way we find that there exists a morphism of the formal moduli

$$H^A(\underline{\mathcal{C}}) \longrightarrow H^B(\underline{\mathcal{C}})$$

of $V = \oplus_{i=1}^r V_i$ as a $k[\Gamma(\underline{\mathcal{C}})] \otimes A$ -module, resp. a $k[\Gamma(\underline{\mathcal{C}})] \otimes B$ -module, consistent with the families. By assumption the above homomorphism induces a surjection

$$\psi_p^* : H_p^A(\underline{\mathcal{C}}) \longrightarrow H_p^B(\underline{\mathcal{C}})$$

By definition of O , we have a commutative diagram,

$$\begin{array}{ccccccc} A & \longrightarrow & O^A(\underline{\mathcal{C}}, \pi) & \hookrightarrow & \text{End}_{H^A}(H_{i,j}^A \otimes V_j) & \longrightarrow & (H_p^A(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)) \\ \psi \downarrow & & O(\psi) \downarrow & & \downarrow & & \downarrow \\ B & \longrightarrow & O^B(\underline{\mathcal{C}}, \pi) & \hookrightarrow & \text{End}_{H^B}(H_{i,j}^B \otimes V_j) & \longrightarrow & (H_p^B(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)). \end{array}$$

where, $O^A(\underline{\mathcal{C}}, \pi)$ and $O^B(\underline{\mathcal{C}}, \pi)$ are the commutants of the actions of $k[\Gamma(\underline{\mathcal{C}})]$ in $S \otimes \text{Hom}_k(V_i, V_j)$, for all quotients S of $(H_p^A(\underline{\mathcal{C}}))$ respectively of $(H_p^B(\underline{\mathcal{C}}))$. The surjectivity of ψ_p^* , together with the commutativity of the diagram defines the morphism $O(\psi)$, and proves (a).

To prove (b), we just have to observe that O^A acts on the H^A -family $(H_{i,j}^A \otimes V_j)$, consistent with the action of A via η_A , and that $k[\Gamma(\underline{\mathcal{C}})] \otimes O$ acts on the $H_p^A(\underline{\mathcal{C}})$ -family $(H_p^A(\underline{\mathcal{C}}) \otimes V)$ consistent with the A-action via the obvious composition,

$$\eta_p(V) : A \longrightarrow (H_p^A(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)).$$

Therefore there must exist morphisms,

$$\begin{aligned} (H_{i,j}^{O^A}) &\xrightarrow{\mu} (H_{i,j}^A) \\ H^{O^A}(\underline{\mathcal{C}}) &\xrightarrow{\mu(\underline{\mathcal{C}})} H^A(\underline{\mathcal{C}}) \end{aligned}$$

consistent with the obvious families. Moreover the composed morphisms

$$\begin{aligned} (H_{i,j}^A) &\rightarrow (H_{i,j}^{O^A}) \xrightarrow{\mu} (H_{i,j}^A) \\ H^A(\underline{\mathcal{C}}) &\rightarrow H^{O^A}(\underline{\mathcal{C}}) \xrightarrow{\mu_p} H_p^A(\underline{\mathcal{C}}) \end{aligned}$$

must be surjections inducing injections on the tangent spaces. It is easy to see that $\mu(\underline{\mathcal{C}})$ induces a surjective homomorphism,

$$H_1^{O^A}(\underline{\mathcal{C}}) \xrightarrow{\mu_0} H_1^A(\underline{\mathcal{C}}).$$

In fact, consider the diagram,

$$\begin{array}{ccccc}
\text{Mor}(H_0^A(\underline{\mathcal{C}}), -) & \longrightarrow & \text{Mor}(H^O(\underline{\mathcal{C}}), -) & \longrightarrow & \text{Mor}(H^A(\underline{\mathcal{C}}), -) \\
\downarrow & \searrow \alpha & \downarrow & & \downarrow \\
\text{Def}_V^{A-\Gamma} & \dashrightarrow & \text{Def}_V^{O-\Gamma} & \xrightarrow{\beta} & \text{Def}_V^{A-\Gamma}
\end{array}$$

Define α by the coposition, i.e. by simply considering the canonical $k[\Gamma] \otimes O$ -structure on $H_0^A(\underline{\mathcal{C}}) \otimes V$, and tensorization. Since the composition of α and β is injective, α is injective. But this implies, by definition and by the the unicity of the modular substratum, that the surjective homomorphism

$$\mu_0 : H^O(\underline{\mathcal{C}}) \longrightarrow H_0^A(\underline{\mathcal{C}})$$

induces a surjective homomorphisms,

$$\mu_0(\underline{\mathcal{C}}) : H^O(\underline{\mathcal{C}})_0 \longrightarrow H_0^A(\underline{\mathcal{C}}).$$

and therefore also a surjection,

$$\mu_0 : H_0^O(\underline{\mathcal{C}}) \longrightarrow H_0^A(\underline{\mathcal{C}}).$$

Consider now the commutative diagram,

$$\begin{array}{ccccc}
A & \longrightarrow & (H_{i,j}^A \otimes \text{Hom}_k(V_i, V_j)) & \longrightarrow & (H_p^A(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)) \\
\eta \downarrow & & \uparrow \mu & & \uparrow \mu_p \\
O(\underline{\mathcal{C}}, \pi) & \longrightarrow & (H_{i,j}^O \otimes \text{Hom}_k(V_i, V_j)) & \longrightarrow & (H_p^O(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)).
\end{array}$$

Since μ_p is consistent with the actions of $k[\Gamma]$, we obtain a cosection

$$\mu : O^O(\underline{\mathcal{C}}, \pi) \longrightarrow O^A(\underline{\mathcal{C}}, \pi)$$

of the morphism

$$\eta_1 : O^A(\underline{\mathcal{C}}, \pi) \longrightarrow O^O(\underline{\mathcal{C}}, \pi).$$

Now use (a) for the case $\psi = \mu$, we obtain a commutative diagram,

$$\begin{array}{ccc}
O^O(\underline{\mathcal{C}}, \pi) & \xrightarrow{\mu} & O^A(\underline{\mathcal{C}}, \pi) \\
\eta_2 \downarrow & & \downarrow \eta_1 \\
O^{OO}(\underline{\mathcal{C}}, \pi) & \xrightarrow{O(\mu)} & O^O(\underline{\mathcal{C}}, \pi)
\end{array}$$

Since, by construction, the composition $\eta_2 \circ O(\mu)$ is an isomorphism, μ must be injective, therefore an isomorphism, proving (b). \square

Non-commutative schemes.

Definition 2.8. Let A be any k -algebra. A finite diagram \underline{c} of A -modules is called a prescheme for A , if the morphism

$$\eta(\underline{c}, \pi) : A \longrightarrow O(\underline{c}, \pi)$$

is an isomorphism. In this case we shall refer to A as the *affine k -algebra* of \underline{c} .

Notice that we have two different O -constructions, $O(p)$, depending on the choice of $H_p(\underline{c})$, for $p = 0, 1$. Since $H_1(\underline{c}) \rightarrow H_0(\underline{c})$ is surjective, it is clear that

$$O(1) \subseteq O(0).$$

We shall, for reasons to be explained later, choose first to work with $O(0)$.

Corollary 2.9. Any finite diagram of right A -modules, \underline{c} , is a prescheme for $O^A(\underline{c}, \pi)$. In particular, if \underline{c} is a diagram of k -vector spaces, then \underline{c} is a diagram of right modules over $A = O^k(\underline{c}, \pi) = O_0(\underline{c}, \pi)$, and, as such, a prescheme for A .

Proof. This follows from the isomorphism

$$O(\eta_A) : O^A(\underline{c}, \pi) \rightarrow O^{O^A}(\underline{c}, \pi)$$

of (2.7). \square

Example 2.10. According to the Generalized Burnside Theorem, if A is any finite dimensional k -algebra, k algebraically closed, the family of simple A -modules V form a (0-dimensional) prescheme for A . In particular, if Λ is a finite partially ordered set, then the set of nodes of Λ , considered as the set of simple $k[\Lambda]$ -modules, is a scheme for $k[\Lambda]$.

Example 2.11. (a): Consider the following trivial example where $A=k$ is a field, $V_1 = k^2$ and $V_2 = k$, and \underline{c} is given by the diagram of right-modules,

$$V_1 \xrightarrow{\phi} V_2$$

where ϕ is the second projection. Obviously all Ext 's vanish, so that

$$H(|\underline{c}|) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

$$(H_{i,j} \otimes V_j) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Moreover $H_0(\underline{c}) = k$, and the maximal ideal $\underline{m} \subset H_0(\underline{c})$ is zero. Therefore,

$$\bar{H}_0(\underline{c}) = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

and

$$(\bar{H}_0(\underline{\mathcal{C}})_{i,j} \otimes V_j) = \begin{pmatrix} V_1 & 0 \\ 0 & V_2 \end{pmatrix}$$

Therefore $O(\underline{\mathcal{C}}, \pi)$ is the commutant algebra of the subalgebra,

$$\text{end}(\underline{\mathcal{C}}) \subseteq \begin{pmatrix} \text{Hom}_k(V_1, V_1) & \text{Hom}_k(V_1, V_2) \\ \text{Hom}_k(V_2, V_1) & \text{Hom}_k(V_2, V_2) \end{pmatrix}$$

generated by ϕ , in the sub k -algebra,

$$\begin{pmatrix} \text{Hom}_k(V_1, V_1) & 0 \\ 0 & \text{Hom}_k(V_2, V_2) \end{pmatrix}$$

which is easily seen to be equal to,

$$O_0(\underline{\mathcal{C}}, \pi) = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

Now it is equally easy to see that V_1 identifies with the second line of

$$O(\underline{\mathcal{C}}, \pi) = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$$

as a right $O(\underline{\mathcal{C}}, \pi)$ -module, and is therefore projective, and that V_2 identifies with the second simple module of $O(\underline{\mathcal{C}}, \pi)$. Trivial calculation gives that

$$\text{Ext}_O^1(V_i, V_j) = 0, \quad \forall i, j.$$

Put $O := O(\underline{\mathcal{C}}, \pi)$. If we consider $\underline{\mathcal{C}}$ as a diagram of O -modules, and repeat the O -construction, we therefore obtain $O^O =: O^{(2)}(\underline{\mathcal{C}}, \pi) = O(\underline{\mathcal{C}}, \pi)$, implying that the diagram $\underline{\mathcal{C}}$ is a prescheme for O , as it must be, see (2.9). Since the $O(\underline{\mathcal{C}}, \pi)$ we have found above is the algebra of the (A_1) diagram,

$$\circ^1 \longleftarrow \circ^2$$

we find that *as O -modules*, the discrete diagram $\mathcal{V} = \{k(1), k(2)\}$ consisting of the two simple O -modules is also a prescheme for O . Notice, however, that as family of k -vector spaces \mathcal{V} is a prescheme for k^2 .

Example 2.12. That the claim of the Corollary (2.9) is not obvious, is seen by classifying the simple schemes of the form,

$$\Gamma = \begin{array}{c} \phi \\ \circ \curvearrowright \end{array}$$

Let $\underline{\mathcal{C}} = \{\phi : V \rightarrow V\}$ be a diagram of k -vector spaces. Assume $\dim_k V = 3$, then according to the Jordan decomposition of ϕ , we obtain the following k -algebras $A = O(\underline{\mathcal{C}}, \pi)$ with $\underline{\mathcal{C}}$ as affine scheme,

- (1) ϕ has 3 different eigenvalues, then $A = k^3$

- (2) ϕ has only two different eigenvalues, then $A = k \times \text{End}_k(k^2)$ or $A = k \times k[\epsilon]$
 (3) ϕ has all eigenvalues equal, then $A = \text{End}_k(k^3)$ or $A = k[x]/(x^3)$ or

$$A = \begin{pmatrix} k & k \cdot \epsilon_{1,2} \\ k \cdot \epsilon_{2,1} & k[\epsilon_{2,2}] \end{pmatrix}$$

where we have the following relations, $\epsilon_{1,2} \cdot \epsilon_{2,1} = 0$, $\epsilon_{1,2} \cdot \epsilon_{2,2} = 0$, $\epsilon_{2,2} \cdot \epsilon_{2,1} = 0$, $\epsilon_{2,1} \cdot \epsilon_{1,2} = \epsilon_{2,2}$

The last algebra is evidently artinian with two simple representations, both of dimension 1. Therefore it has an affine scheme consisting of the discrete diagram consisting of those two A-modules. As such, it has infinitesimal incidences given by the family $\{\epsilon_{i,j}\}$. By (2.9) A is also the affine algebra of the scheme $\underline{\mathcal{C}}$, with quiver Γ .

Example 2.13. NB! According to Example (2.4) if A is any reduced finite type commutative k -algebra, and if k is algebraically closed, $A \simeq O^k(\text{Spec}(A), \pi)$. However $\text{Spec}(A)$ is usually infinite. Therefore we cannot conclude from Corollary (2.9), that $\text{Spec}(A)$ is a prescheme for A . In fact, $\text{Spec}(A)$ is in general not a prescheme for A , since $O(\text{Spec}(A), \pi)$ may well be non-commutative.

Example 2.14. Consider the special case of Remark (2.6) (c), where A is a local k -algebra, and $\phi : A \rightarrow k$ is the canonical homomorphism of A onto its residue field. Since there is a surjective homomorphism,

$$H_0(\{\phi\}) \longrightarrow H_{2,2} = \hat{A}.$$

and since the completion map ρ_x is a lifting,

$$\Phi_{1,2} : \hat{A} \otimes A \rightarrow H_0(\{\phi\}) \otimes k = \hat{A}$$

of $\phi_{1,2}$, we observe that (14) implies that $\alpha_{1,1}$ is the right multiplication by $a = \alpha_{1,1}(1) \in A$. Moreover $\alpha_x = \rho_x(a)$ and (15) is automatically satisfied, since $\Phi_{1,2}$ is A -linear. Therefore,

$$O(\{\phi\}, \pi) = \left\{ \begin{pmatrix} \alpha & 0 \\ \alpha_{2,1} & \alpha_x \end{pmatrix} \mid \alpha \in A, \alpha_x = \rho_x(\alpha), \alpha_{2,1} \in H_{2,1} \otimes \text{Hom}_k(k, A) \right\}.$$

In particular, the natural morphism

$$\eta : A \rightarrow O(\{\phi\}, \pi)$$

is injective, and an isomorphism provided $\text{Ext}_A^1(k, A)^\phi = \text{Ext}_A^1(k, A)$.

The k -algebra of preobservables of a diagram of modules is not, however, properly functorial with respect to inclusions between diagrams. The problem is the following: Let $\mathcal{L}_0 \subseteq \mathcal{C}$ be a finite subdiagram. There is a corresponding inclusion of quivers, $\Gamma(\mathcal{L}_0) \subseteq \Gamma(\mathcal{C})$, inducing a homomorphism of k -algebras,

$$k[\Gamma(\mathcal{L}_0)] \longrightarrow k[\Gamma(\mathcal{C})].$$

This implies the existence of commutative diagrams,

$$\begin{array}{ccc}
(H(|\underline{\mathcal{C}}|)_{i,j} \otimes \text{Hom}_k(V_i, V_j)) & \longrightarrow & (H(|\underline{\mathcal{C}}_0|)_{i,j} \otimes \text{Hom}_k(V_i, V_j)) \\
\downarrow & & \downarrow \\
(H_0(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j)) & & (H_0(\underline{\mathcal{C}}_0) \otimes \text{Hom}_k(V_i, V_j)) \\
\uparrow & & \uparrow \\
k[\Gamma] & \longleftarrow & k[\Gamma_0]
\end{array}$$

We would now like to conclude that this induces a natural morphism

$$O(\underline{\mathcal{C}}, \pi) \longrightarrow O(\underline{\mathcal{C}}_0, \pi)$$

since there are fewer conditions to be satisfied in the definition of the right hand algebra. However, the diagram above shows that this is not obvious. An element in $O(\underline{\mathcal{C}}, \pi)$ certainly commutes with the action of $k[\Gamma_0]$ in $(H_0(\underline{\mathcal{C}}) \otimes \text{Hom}_k(V_i, V_j))$ but not necessarily with the action of $k[\Gamma_0]$ in $(H_0(\underline{\mathcal{C}}_0) \otimes \text{Hom}_k(V_i, V_j))$. But we are interested in the smallest k -algebra O extending A , and preserving both the diagram and the system of extensions of extensions of the objects of the diagram. Therefore we define the refined k -algebra of observables of the diagram $\underline{\mathcal{C}}$, as,

Definition 2.15. The k -algebra of *observables* $\mathcal{O}(\underline{\mathcal{C}}, \pi)$ of the finite diagram $\underline{\mathcal{C}}$, is the subalgebra of $(H_{i,j}(|\underline{\mathcal{C}}|) \otimes \text{Hom}_\pi(V_i, V_j))$ mapping into

$$\mathcal{O}(\underline{\mathcal{C}}_0, \pi) \subseteq (H(\underline{\mathcal{C}}_0) \otimes \text{Hom}_k(V_i, V_j))$$

for all subdiagrams $\underline{\mathcal{C}}_0 \subseteq \underline{\mathcal{C}}$.

It is easy to see that the k -algebra of observables is a contravariant functor on the ordered set of subdiagrams of a given diagram.

We are now able to extend the definition of observables to infinite diagrams. For any given diagram $\underline{\mathcal{C}}$, for which there exists a natural action

$$\eta(|\underline{\mathcal{C}}|, \pi) : A \longrightarrow O(|\underline{\mathcal{C}}|, \pi),$$

we define,

$$\mathcal{O}(\underline{\mathcal{C}}, \pi) = \varprojlim_{\underline{\mathcal{C}}_0 \subseteq \underline{\mathcal{C}}} \mathcal{O}(\underline{\mathcal{C}}_0, \pi)$$

where $\underline{\mathcal{C}}_0$ runs through all finite subdiagrams of $\underline{\mathcal{C}}$. Clearly

$$\mathcal{O}(\underline{\mathcal{C}}, \pi) \subseteq O(|\underline{\mathcal{C}}|, \pi)$$

and there is a natural homomorphism of k -algebras,

$$\eta(\underline{\mathcal{C}}, \pi) : A \longrightarrow \mathcal{O}(\underline{\mathcal{C}}, \pi)$$

the obvious limit of the family of morphisms $\eta(\underline{\mathcal{C}}_0, \pi)$, where $\underline{\mathcal{C}}_0$ runs through all finite subdiagrams of $\underline{\mathcal{C}}$.

Definition 2.16. The diagram \underline{c} will be called a *prescheme* for A , if $\eta(\underline{c}, \pi)$ is an isomorphism.

Notice that if the finite diagram \underline{c} is a prescheme in the sense of (2.8) then it is also a prescheme in the sense of (2.16). Therefore, any finite diagram of A -modules is necessarily a prescheme for $O(\underline{c}, \pi)$. This is, however, not the case for infinite diagrams, see (2.13).

Finally, we arrive at the notion of *structure sheaf* \mathcal{O}_π . For every finite subdiagram \underline{c}_0 of \underline{c} , consider the natural morphism,

$$\kappa(\underline{c}_0) : \mathcal{O}(\underline{c}, \pi) \rightarrow (H_0(\underline{c}_0) \otimes \text{Hom}_k(V_i, V_j))$$

and consider the two-sided ideal $\mathfrak{n} \subset \mathcal{O}(\underline{c}, \pi)$, defined by

$$\mathfrak{n} = \bigcap_{\underline{c}_0 \subseteq \underline{c}} \ker \kappa(\underline{c}_0)$$

Here \underline{c}_0 runs through all finite subdiagrams of \underline{c} .

Definition 2.17. (i) In the above situation, put,

$$\mathcal{O}_\pi(\underline{c}) = \mathcal{O}(\underline{c}, \pi) / \mathfrak{n}$$

(ii) Let A be any k -algebra. A diagram \underline{c} of A -modules is called a *scheme* for A , if the canonical morphism

$$\eta(\underline{c}, \pi) : A \longrightarrow \mathcal{O}_\pi(\underline{c})$$

is an isomorphism. In this case we shall call A the *affine k -algebra* of \underline{c} .

(iii) \mathcal{O}_π is a presheaf on the ordered set of subdiagrams of a given diagram \underline{c} , for which there exists a natural action

$$\eta(|\underline{c}|, \pi) : A \longrightarrow \mathcal{O}(|\underline{c}|, \pi).$$

(iv) We shall consider the objects V_i of \underline{c} as *points*, and the morphisms as *incidences* in our *geometry*.

(v) Let for a point V_i of \underline{c} , $H_0(V_i)$ be the modular substratum of the local moduli $H(V_i)$ of V_i , as A -module. Then there exists a natural localization morphism,

$$\mathcal{O}_\pi(\underline{c}) \rightarrow H_0(V_i) \otimes \text{End}_k(V_i)$$

This is in tune with the general setup, see the Introduction. Note that if a diagram \underline{c} of A -modules is a prescheme for A , then it is not necessarily a scheme for A . However, we have the following obvious,

Lemma 2.18. (i) The diagram,

$$\begin{array}{ccc} \mathcal{O}(\underline{c}, \pi) & \longrightarrow & \mathcal{O}_\pi(\underline{c}) \\ \uparrow & & \downarrow \\ A & \longrightarrow & \mathcal{O}^k(\underline{c}, \pi) \end{array}$$

commutes.

(ii) If \underline{c} is a prescheme for A and the homomorphism $A \rightarrow O_0(\underline{c}, \pi) = O^k(\underline{c}, \pi)$ is an isomorphism, then \underline{c} is a scheme for A .

Now, by (2.17) (iv), there exist for every object V_i of \underline{c} , a natural morphism,

$$\mathcal{O}_\pi(\underline{c}) \rightarrow H_0(V_i) \otimes \text{End}_k(V_i)$$

This enables us to define a natural *finest topology* on \underline{c} consisting of *open* subdiagrams.

Definition 2.19. A subdiagram \underline{c}_1 of \underline{c} is *open* in the *finest topology*, if for every object V_i of \underline{c}_1 the images of the morphisms,

$$\mathcal{O}_\pi(\underline{c}_1) \rightarrow H_0(V_i) \otimes \text{End}_k(V_i) \leftarrow \mathcal{O}_\pi(\underline{c})$$

coincide.

Depending on the situation, the usefull topologies will normally be much coarser. We may also generalize the notion of topology to include the analogue of a Grothendieck topology. This seems to give us a satisfactory globalization procedure for schemes.

So far we have defined for each diagram of A -modules \underline{c} a topology of subdiagrams, and a presheaf of observables defined on this topology. Starting with the *geometry*, i.e. the collection of points and incidences, we have thus defined, and studied, the algebra $\mathcal{O}_\pi(\underline{c})$ of *operators parametrizing the geometry*. When \underline{c} is finite, the fact that $O(-, \pi)$ is a closure operator implies that the pair $(O := O(\underline{c}, \pi), \underline{c})$ is a prescheme, i.e. such that $\eta : O \rightarrow O^O(\underline{c}, \pi)$ is an isomorphism. If also $\eta_0 : O \rightarrow O^k(\underline{c}, \pi)$ is an isomorphism it follows from (2.18) that (O, \underline{c}) is a scheme.

Now, if we start with a k -algebra A , how do we find its *scheme*? Obviously, there is no such thing, in this field, as a unique scheme associated to a given k -algebra. However we have the following examples:

Examples 2.20. Let A be any k -algebra, and let $\underline{c} := \text{Ind}(A)$ be the diagram consisting of the essential (i.e. generating all others) morphisms between (all) the indecomposable A -modules. Suppose A is a sum of a finite number of indecomposables. Using the fact that the only k -linear endomorphisms of A that are right A -linear, are the left multiplication by elements of A , we easily prove that $\eta_0 : A \rightarrow O_0(\text{Ind}(A), \pi)$ is an isomorphism. Since $\text{Ind}(A)$ is, essentially, a finite diagram, we may use the closure property of \mathcal{O} and prove that $\text{Ind}(A)$ is a prescheme for A and therefore, by (2.18), also a scheme.

(ii) Consider a discrete diagram $\underline{c} = \mathcal{V}$ of A -modules. There is, by definition of \mathcal{O}_π , a homomorphism,

$$\mathcal{O}_\pi(\mathcal{V}) \longrightarrow (H_0(V) \otimes \text{Hom}_k(V_i, V_j)).$$

The tangent space of the image is

$$(E_{i,j} \otimes \text{Hom}_k(V_i, V_j)),$$

where the $E_{i,j}$ are defined by,

$$(E_{i,j}) = (Ext_A^1(V_i, V_j))^{End_A(V)}$$

(iii) In particular, if \mathcal{V} is the family of all simple right modules of an artinian k -algebra A , with $k = \bar{k}$, then $End_A(V)$ is trivial, and we obtain,

$$A \simeq \mathcal{O}_\pi(\mathcal{V})$$

so \mathcal{V} is a scheme for A .

(iv) Going back to Example (2.14), where A was a local k -algebra with $\phi : A \rightarrow k$ as the residue map, we found that when $\phi_* : Ext_A^1(k, A) \rightarrow Ext_A^1(k, k)$ vanished, the natural morphism,

$$\eta : A \rightarrow \mathcal{O}(\{\phi\}, \pi)$$

is an isomorphism. It is now clear that, in all cases,

$$\eta : A \rightarrow \mathcal{O}_\pi(\{\phi\})$$

is an isomorphism, implying that $\{\phi\}$ is a scheme for A .

Consider for any finite type k -algebra the discrete diagram, $Simp(A)$, consisting of all the simple A -modules. The corresponding morphism,

$$\eta(Simp(A)) : A \longrightarrow \mathcal{O}(Simp(A), \pi)$$

is an isomorphism for finite dimensional k -algebras when k is algebraically closed. In the general case this is, however, obviously not true. In fact, when A is commutative, it is easy to see that

$$\mathcal{O}(Simp(A), \pi) \simeq \prod_{\mathfrak{m}} \hat{A}_{\mathfrak{m}}$$

where \mathfrak{m} runs through all maximal ideals of A . To obtain a General Burnside Theorem, see [17], we must introduce some *generic points* into $Simp(A)$, with the purpose of singeling out an algebraic subring of $\mathcal{O}(Simp(A), \pi)$. In fact, we should include modules like the indecomposable components of A , as in (2.20), i.e. some *generating projectives*, or just A , see sections (3) and (4).

Example 2.21: The Hairy Line. Consider the k -algebra of matrices,

$$A = \begin{pmatrix} k[y] & k[y] \\ 0 & k[y] \end{pmatrix}$$

The scheme $Ind(A)$ contains the two projectives,

$$V_{1,2} = \begin{pmatrix} k[y] & k[y] \\ 0 & 0 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 0 & 0 \\ 0 & k[y] \end{pmatrix}$$

such that $A = V_{1,2} \oplus V_2$. Therefore $Ind(A)$ is a scheme for A . Notice that there is an incidence, i.e. a morphism of (right) A -modules $V_2 \rightarrow V_{1,2}$, the cokernel of which is the indecomposable A -module,

$$V_1 = \begin{pmatrix} k[y] & 0 \\ 0 & 0 \end{pmatrix}$$

Notice also that $A_i := V_i$, $i=1,2$ are quotient algebras of A both isomorphic to the polynomial k -algebra $k[y]$. The closed points of $Ind(A)$ are the different simple representations of A , that is, the different closed points of $Spec(A_i)$, for $i=1,2$. Thus we find that the closed points correspond to the points of two different affine lines, L_1 and L_2 , both (canonically) isomorphic to $Spec(k[y])$. However, while there are no ordinary incidences between these points, there are infinitesimal incidences between pairs of *equal* points $(p, p) \in L_1 \times L_2$. In fact if $p \in L_1$, $q \in L_2$, then

$$\begin{aligned} Ext_A^1(k(p), k(q)) &= 0, \text{ if } p \neq q, = k \text{ if } p = q \\ Ext_A^1(k(q), k(p)) &= 0 \text{ for all } p \in L_1, q \in L_2 \end{aligned}$$

The picture of this is an ordinary line L_2 with *hairs*, corresponding to the points of the other line L_1 , stuck into the first line at the corresponding point.

Example 2.22: Espaces quantiques de Connes. Let A be a commutative k -algebra, k algebraically closed. Consider an algebraic equivalence relation, $\underline{R} = Spec(R)$ on the affine scheme $X = Spec(A)$. It corresponds to the affine diagram,

$$A \begin{array}{c} \xrightarrow{i_1} \\ \xrightarrow{i_2} \end{array} A \otimes A \xrightarrow{\rho} R \longrightarrow A$$

with the obvious relations. Consider the morphisms $\tau_i = i_i \circ \rho$, and let

$$Spec(A : R) = \{R/\tau_1(\mathfrak{p})R \mid \mathfrak{p} \in Spec(A)\}$$

be the diagram consisting of the objects $R/\mathfrak{p}R := R/\tau_1(\mathfrak{p})R$ considered as A -modules via τ_2 , and with the obvious morphisms. Let $x \in X$ be a closed point, corresponding to the maximal ideal \underline{m}_x of A , then the composition,

$$\kappa_x : A \xrightarrow{i_2 \circ \rho} R \xrightarrow{\pi_x} R/\underline{m}_x R$$

identifies $Spec(R/\underline{m}_x R)$ with the equivalence class \bar{x} of $x \in X = Spec(A)$. In particular, if $x \sim_R y$ then

$$R/\underline{m}_x R = R/\underline{m}_y R$$

as objects of $Spec(A : R)$.

Now compute

$$O_0 := O_0(Spec(A : R), \pi)$$

It is clear that, $\alpha \in O_0$ is a family,

$$\alpha = \{\alpha_{\mathfrak{p}} \in End_k(R/\tau_1(\mathfrak{p})R) \mid \pi_{\mathfrak{q}}^{\mathfrak{p}} \circ \alpha_{\mathfrak{q}} = \alpha_{\mathfrak{p}} \circ \pi_{\mathfrak{q}}^{\mathfrak{p}}, \forall \mathfrak{p}, \mathfrak{q} \in Spec(A)\}$$

where $\pi_{\mathfrak{q}}^{\mathfrak{p}} : R/\tau_1(\mathfrak{p})R \rightarrow R/\tau_1(\mathfrak{q})R$ is the obvious morphism corresponding to an inclusion $\mathfrak{p} \subseteq \mathfrak{q}$. Since each one of the $\pi_{\mathfrak{q}}^{\mathfrak{p}}$ is surjective, it is easy to see that α is determined by $\alpha_0 \in \text{End}_k(R)$ and the components $\alpha_x \in \text{End}_k(R/\underline{m}_x R)$. Moreover, since for $x \sim y$, $R/\underline{m}_x R = R/\underline{m}_y R$, as objects, it is clear that α_x only depends upon the equivalence class \bar{x} of x . Each $\alpha_{\bar{x}}$ is a k -linear endomorphism. Suppose the equivalence classes of R are finite reduced, i.e. $R/\underline{m}_x R \simeq \bigoplus_{z \in \bar{x}} k(z)$, then $\alpha_{\bar{x}}$ is a matrix

$$\alpha_{\bar{x}} = (\alpha(z, z')), \quad z, z' \in \bar{x}, \quad \alpha(z, z') \in k.$$

Moreover, in this case, given $\alpha, \alpha' \in \mathcal{O}_0$ then $\alpha = \alpha'$ if and only if $\alpha_{\bar{x}} = \alpha'_{\bar{x}}$ for all closed points $x \in X$. This follows immediately from the defining relations, $\pi_x(\alpha(a)) = \alpha_{\bar{x}}(\pi_x(a))$. This means that α is determined as a *function* on $X \times X$ with $\alpha(z, z') = 0$ unless $z \sim z'$, and with addition defined as for functions, but with multiplication defined by the matrix nature of each $\alpha_{\bar{x}}$, i.e. $(\alpha\alpha')(x, y) = \sum_{z \in \bar{x} = \bar{y}} \alpha(x, z)\alpha'(z, y)$. But this is the way Connes defines *les espaces quantiques*, see [7].

Functoriality of $\mathcal{O}(-, \pi)$ and \mathcal{O}_{π} . The $\mathcal{O}(-, \pi)$, and therefore also \mathcal{O}_{π} , are presheaves on the ordered set of subdiagrams of some given diagram. Fixing the k -algebra A , and consider a diagram of right A -modules \underline{c} . Assume $B := \mathcal{O}^A(\underline{c}, \pi) = \mathcal{O}^B(\underline{c}, \pi)$, such that (B, \underline{c}) is a prescheme. It is reasonable to call such preschemes *A-preschemes* or *preschemes relative to A*. Therefore, the affine k -algebra of A -schemes is a contravariant functor on the category of A -schemes with respect to inclusions. We may define a morphism Φ of the scheme \underline{c}_1 in \underline{c}_2 , as a diagram \underline{c}_{Φ} , containing both \underline{c}_1 and \underline{c}_2 , and such that the last inclusion induces an isomorphism

$$\mathcal{O}(\underline{c}_{\Phi}, \pi) \simeq \mathcal{O}(\underline{c}_2, \pi)$$

There is an obvious problem related to composing morphisms, which can be overcome in some interesting cases, for example in the commutative case.

Infinitesimal structures on schemes. Let \underline{c} be a diagram of A -mod, and consider a point $x = V_i$. We would like to be able to talk about vectorfields, their values at points, energy operators and time etc. as in quantum mechanics. We start with the following,

Definition 2.23. Given a point $x = V_i \in \underline{c}$, we put

$$\begin{aligned} T_x &:= \text{Ext}_A^1(V_i, V_i)^{\Gamma} : \\ &= \{ \xi \in \text{Ext}_A^1(V_i, V_i) \mid \forall p \exists \xi_p \in \text{Ext}_A^1(V_p, V_p) \text{ such that } \forall \phi := \phi_{i,p} : V_i \rightarrow V_p, \\ &\quad \phi_*(\xi) = \phi^*(\xi_p) \text{ and } \forall \phi := \phi_{p,i} : V_p \rightarrow V_i, \phi_*(\xi_p) = \phi^*(\xi) \} \end{aligned}$$

and we shall call it the *tangent space* of \underline{c} at x .

There is a canonical map

$$(12) \quad \kappa_x : \text{Der}_k(A, A) \longrightarrow T_x$$

the compositions of the natural maps,

$$\text{Der}_k(A, A) \longrightarrow \text{Der}_k(A, \text{End}_k(V))$$

the surjection

$$Der_k(A, End_k(V)) \longrightarrow Ext_A^1(V, V) = \oplus_{i,j} Ext_A^1(V_i, V_j)$$

and the projection onto $Ext_A^1(V_i, V_i)$. Recall that the tangent space T_p of $H_p(\underline{\mathcal{C}})$ is given by,

$$T_1 = Ext_A^1(V, V)^\Gamma \text{ and } T_0 = (Ext_A^1(V, V)^\Gamma)^{End_k[\Gamma] \otimes A(V)}$$

and see that there are natural homomorphisms,

$$Der_k(A, A) \rightarrow T_0 \rightarrow T_1$$

For every $\delta \in Der_k(A, A)$, let $\delta(x) \in T_x$, be the image $\kappa_x(\delta)$ in T_x . Thus $Der_k(A, A)$ is a k -vectorspace of *vector fields* defined on $\underline{\mathcal{C}}$. In particular if $E \in A$ is some element, then the k -linear map $adE : A \rightarrow A$ defined by $adE(a) = Ea - aE$, is a k -derivation of A . Obviously, $adE(x) = 0$ for every point x of $\underline{\mathcal{C}}$. Thus, operators of this type (like time evolutions), i.e. $ad(E)$, although they act on the observables, leave the *points* fixed.

Suppose now that $\underline{\mathcal{C}}$ is a scheme for A , then;

Definition 2.24. Given a point $x=V_i$ of $\underline{\mathcal{C}}$, we shall say that x is smooth if the map κ_x is surjective. A prescheme $(A, \underline{\mathcal{C}})$ is smooth if all the points are smooth.

This generalizes the classical notion of smoothness in algebraic geometry. Notice that for any k -algebra A , k algebraically closed, and for any finite subset $\{V_i\}_i$ of finite dimensional objects of $Simp(A)$, the morphisms,

$$A \longrightarrow End_k(V_i), \text{ and } Der_k(A, Hom_k(V_i, V_j)) \rightarrow Ext_A^1(V_i, V_j)$$

are surjectives. This proves immediately that the *Burnside morphism*,

$$A \longrightarrow O(\{V_i\}_i, \pi)$$

is locally dense, in the sense that it is injective at the tangent level. When all simple modules of A are finite dimensional, and sit in one of a finite number of algebraic families of such, we conjecture that there is a finite number of generic points, which added to the diagram $Simp(A)$ produces a diagram $Simp^*(A)$, which is a scheme for A , i.e. such that the Burnside morphism becomes an isomorphism. Adding the *generic points* produces an algebraization of $\mathcal{O}(Simp(A), \pi)$.

3 The commutative case.

The main Theorem. To show that the non-commutative algebraic geometry, introduced above, is a bona fide extension of classical algebraic geometry, one would be tempted to prove that, for commutative k -algebras A ,

$$(S) \quad \eta(Spec(A), \pi) : A \longrightarrow \mathcal{O}_\pi(Spec(A))$$

is an isomorphism. This is, however, not reasonable, see the Introduction. The problem is that $Spec(A)$ contains too many points. The closed points are special in that $Spec(A)$ is the moduli space for such. The others, the non closed points, are not treated as bona fide points, but as *generic points* for subschemes. We would like to include as few as possible such generic points, and therefore we add only the projective generator of $\underline{\mathcal{C}}$, i.e. A . We have already proved the essentials of the following, see (2.20),(iv).

Theorem 3.1. *Let A be any commutative k -algebra, essentially of finite type, with k algebraically closed. Let $\text{Simp}^*(A)$ be the diagram $\text{Simp}(A)$ augmented by the generic point A . Then the canonical morphisms of k -algebras*

$$\eta(\text{Simp}^*(A), \pi) : A \rightarrow \mathcal{O}_\pi(\text{Simp}^*(A))$$

is an isomorphism.

Proof. For every closed point $x \in \text{Simp}(A)$ consider the corresponding homomorphisms of A -modules,

$$\phi_x : A \rightarrow k(x)$$

and use the Remark (2.6), to see that the versal deformation of ϕ_x is the canonical morphism of k -algebras

$$\Phi_x : A \rightarrow H_{x,x} \otimes \text{End}_k(k(x)) = \hat{A}_x.$$

If $\alpha \in \text{End}_k(A)$ and $\alpha_x \in \text{End}_{H_{x,x}}(H_{x,x} \otimes \text{End}_k(k(x)))$ commute via the action of Φ_x , then it is easy to see that the composition $(\alpha - R_{\alpha(1)}) \circ \Phi_x = 0$, where $R_{\alpha(1)}$ is the right multiplication by $\alpha(1)$. Since this is true for all x and all $x \in \text{Simp}(A)$ we obtain that α is the right multiplication of some element $\alpha(1)$, proving our Proposition. \square

We have also to show that the \mathcal{O}_π -construction applied to the obvious subdiagrams of the category $\mathcal{O}_X - \text{Mod}$, where X is a k -scheme, gives us a globalization procedure.

According to paragraph (2) above, the globalization procedure is a consequence of the functorial properties of \mathcal{O}_π . To fix the ideas let us consider the following example.

Example 3.3. Blowing ups. Let $A = k[x, y]$ and consider the A -module $V = (x, y)$, i.e. the maximal ideal of A . The diagram of A -modules, $\text{Simp}^*(A - V)$ is, by definition, the diagram consisting of V together with all simple quotients of V . These correspond to all points of $\mathbf{A}^2 = \text{Spec}(A)$ different from $(0,0)$, together with all tangent lines through $(0,0)$ in \mathbf{A}^2 . Now consider $x \in V$ and let $D(x)$ be the diagram obtained from $\text{Simp}^*(A - V)$ by localizing, i.e. removing the points where x becomes zero, and inverting the multiplicative action by x . In particular, in $D(x)$ we have all points of \mathbf{A}^2 minus the y -axis, but preserving all tangent lines through $(0,0)$, except the x -axis. Now we compute, and we find,

$$\mathcal{O}(\text{Simp}^*(A - V), \pi) = A, \quad \mathcal{O}(D(x), \pi) = k[y, y/x]$$

as we should, proving that the scheme $(\text{Simp}^*(A - V), \mathcal{O})$ is the blow up of the origin in \mathbf{A}^2

Notice that in this paragraph we have assumed that A is a commutative k -algebra essentially of finite type on an algebraically closed field. The extension of the theory to include schemes on general base rings, seems difficult.

4 Invariant theory and moduli.

Consider a commutative finite type reduced k -algebra A , k algebraically closed. Suppose that there is a Lie-algebra \mathfrak{g} of vectorfields (i.e. derivations), acting on A . Consider the category, \underline{C} of $A - \mathfrak{g}$ -modules, i.e. A -modules with integrable \mathfrak{g} -covariant derivations. In this category we define the trivialization functor,

$$\pi_{\mathfrak{g}} : \underline{C} \longrightarrow k - \text{mod}$$

by

$$\pi_{\mathfrak{g}}(V) = H^0(\mathfrak{g}, V)$$

Recall that when \mathfrak{g} is semisimple then $\pi := \pi_{\mathfrak{g}}$ is exact. Moreover, Ext , in this category, is then simply the \mathfrak{g} -invariants of the Ext in $A\text{-mod}$. The exactness of $\pi_{\mathfrak{g}}$, is, however, not necessary for the constructions we have in mind. Given any diagram \underline{c} of \underline{C} , say $\text{Simp}^*(A - \mathfrak{g})$, corresponding to the ordered set of the minimal and maximal \mathfrak{g} -invariant prime ideals of A , we may consider the ring of observables $O(\underline{c}, \pi_{\mathfrak{g}})$. In particular, we pose,

Definition 4.1. The quotient of $\text{Spec}(A)$ with \mathfrak{g} , denoted by $\text{Spec}(A)/\mathfrak{g}$, is the presheaf of k -algebras of observables,

$$O(\text{Simp}^*(A - \mathfrak{g}), \pi_{\mathfrak{g}})$$

Notice that in the above formalism, I might have considered, instead of the Lie-algebra \mathfrak{g} , any group G , and for that matter, any other reasonable superstructure on the category of A -modules.

That this invariant theory fits with the classical invariant theory, is shown by the following result,

Theorem 4.2. *Let A be any irreducible and reduced commutative k -algebra of finite type, k algebraically closed, and \mathfrak{g} a semi-simple Lie-algebra of vectorfields (i.e. derivations), acting on A . Assume that the geometric quotient of $\text{Spec}(A)$ with \mathfrak{g} exists, (and is affine). Then it coincides with the $\text{Spec}(A)/\mathfrak{g}$, defined above.*

Proof. By assumption, the diagram $\text{Simp}^*(A^{\mathfrak{g}})$ induces the diagram $\text{Simp}^*(A - \mathfrak{g})$. Moreover the trivialization $\pi_{\mathfrak{g}}$ maps the diagram $\text{Simp}^*(A - \mathfrak{g})$ onto the diagram $\text{Simp}^*(A^{\mathfrak{g}})$, or rather, to the image of this diagram under the canonical trivialization π . But then the exactness of $\pi_{\mathfrak{g}}$ and the smoothness of the morphism of the geometric quotient, proves that the formal moduli of the family $|\text{Simp}^*(A - \mathfrak{g})|$ in the category of $A - \mathfrak{g}\text{-mod}$ is isomorphic to the corresponding formal moduli of the family $|\text{Simp}^*(A - \mathfrak{g})|$ in the category of $A^{\mathfrak{g}}\text{-mod}$. Since the trivializations coincide, the Theorem (3.1) shows that

$$O(\text{Simp}^*(A - \mathfrak{g}), \pi_{\mathfrak{g}}) \cong A^{\mathfrak{g}}$$

which is exactly what we wanted. \square

5 Tensor products and quantum groups. Let $\underline{\mathcal{C}}$ be a subcategory of $\mathbf{A}\text{-mod}$, with a trivializing functor π . Suppose given a tensor product on the category $\underline{\mathcal{C}}$, i.e. bi-functor

$$\otimes : \underline{\mathcal{C}} \times \underline{\mathcal{C}} \longrightarrow \underline{\mathcal{C}}$$

which is a faithful imbedding, consistent with π , with some extra structure. In particular there should exist natural isomorphisms,

$$\alpha_{-, -, -} : ((- \otimes -) \otimes -) \simeq (- \otimes (- \otimes -))$$

satisfying the Mac Lane pentagon,

$$\begin{aligned} id_X \otimes \alpha_{Y, Z, W} \circ \alpha_{X, Y \otimes Z, W} \circ \alpha_{X, Y, Z} \otimes id_W \\ = \alpha_{X, Y, Z \otimes W} \circ \alpha_{X \otimes Y, Z, W} \end{aligned}$$

Consider the exact functor,

$$\Delta : \underline{\mathcal{C}} \rightarrow \underline{\mathcal{C}} \times \underline{\mathcal{C}},$$

defined by $\Delta(V) = V \times V$. Then, since all of these functors are imbeddings, there exist homomorphisms of k -algebras,

$$(1) \quad \mathcal{O}(\underline{\mathcal{C}}, \pi) \rightarrow \mathcal{O}(\underline{\mathcal{C}} \otimes \underline{\mathcal{C}}, \pi) \simeq \mathcal{O}(\underline{\mathcal{C}} \times \underline{\mathcal{C}}, \pi) \rightarrow \mathcal{O}(\underline{\mathcal{C}}, \pi)$$

such that,

$$\mathcal{O}(\underline{\mathcal{C}} \times \underline{\mathcal{C}}, \pi) \simeq \mathcal{O}(\underline{\mathcal{C}}, \pi) \otimes \mathcal{O}(\underline{\mathcal{C}}, \pi)$$

and the last morphism of (1) is the multiplication morphism of the k -algebra $\mathcal{O}(\underline{\mathcal{C}}, \pi)$.

The Mac Lane pentagon guarantees that the first morphism of (1) becomes an associative co-algebra structure on $\mathcal{O}(\underline{\mathcal{C}}, \pi)$. Clearly any extra functorial *symmetry* one may want to consider on $\underline{\mathcal{C}}$, will show up in the corresponding k -algebra $\mathcal{O}(\underline{\mathcal{C}}, \pi)$.

6 Examples.

The non-commutative projective line. Let $A = k[x_0, x_1]$, and consider the usual k^* -action. We shall compute the space $\text{Spec}(A)/k^*$. The subcategory $\text{Simp}(A\text{-}k^*)$ of $A\text{-}k^*$ -modules consists of the origin V_3 , the lines through the origin $V_2(l)$, and the generic point V_1 . The trivializing functor (see section (4)),

$$\pi : A\text{-}k^*\text{-mod} \longrightarrow k\text{-mod}$$

has the values,

$$\pi(V_1) = k, \quad \pi(V_2(l)) = k, \quad \pi(V_3) = k$$

Therefore there are no π -incidences, and the non-commutative orbit space is given by the hull of the deformation functor, i.e. by $(H_{i,j})$. Since $H^p(k^*, -) = 0$ for $p \geq 1$, we may use the Proposition (1.2), and we obtain,

$$\text{Ext}_{A\text{-}k^*}(V_i, V_j) = \text{Ext}_A(V_i, V_j)^{k^*}.$$

It is easy to compute the different ext-groups, we find:

$$\begin{aligned}
Ext_A^1(V_i, V_j) &= 0, \text{ for } i=1, j=1,2,3. \\
Ext_A^1(V_2(l), V_1) &= V_2(l) = A/(\alpha x_0 + \beta x_1) \\
Ext_A^1(V_2(l), V_2(l)) &= V_2(l) \\
Ext_A^1(V_2(l), V_2(l')) &= 0 \text{ if } l \neq l' \\
Ext_A^1(V_2(l), V_3) &= V_3 = k \\
Ext_A^1(V_3, V_1) &= 0 \\
Ext_A^1(V_3, V_2(l)) &= V_3 = k \\
Ext_A^1(V_3, V_3) &= k^2
\end{aligned}$$

Using the results of paragraph 1.2. we obtain for the invariants

$$\begin{aligned}
Ext_A^1(V_i, V_j)^{k^*} &= 0, \text{ for } i=1, j=1,2,3. \\
Ext_A^1(V_2(l), V_1)^{k^*} &= k \text{ represented by } \xi = l \\
Ext_A^1(V_2(l), V_2(l))^{k^*} &= k \text{ represented by } \xi = l \\
Ext_A^1(V_2(l), V_2(l'))^{k^*} &= 0 \text{ if } l \neq l' \\
Ext_A^1(V_2(l), V_3)^{k^*} &= 0 \\
Ext_A^1(V_3, V_1)^{k^*} &= 0 \\
Ext_A^1(V_3, V_2(l))^{k^*} &= 0 \\
Ext_A^1(V_3, V_3)^{k^*} &= 0
\end{aligned}$$

The corresponding quotient becomes the infinite matrix algebra of the form,

$$Spec(A)/k^* := O(Simp(A - k^*), \pi) = \begin{pmatrix} k & 0 & 0 \\ k[[t_2(l)]]t_{2,1} & k[[t_2(l)]] & 0 \\ 0 & 0 & k \end{pmatrix}$$

where l runs through all the points in the ordinary projective line. We observe that the special point, corresponding to the isolated orbit, i.e. the origo, stays isolated, even infinitesimally. There are, however, adjacencies between the formal points corresponding to the lines through the origo, and the *generic* point corresponding to the generic point of the ordinary projective line.

Suppose that we localize, say in x_0 , i.e. that we restrict to the

$$Spec(A_{\{x_0\}} - k^*) = \{V_1 = A_{\{x_0\}}, V_2(l) = A_{\{x_0\}}/(l)\}$$

then we find,

$$\pi(V_1) = k[x_1/x_0], \pi(V_2(l)) = k$$

and therefore the π -incidences,

$$\pi(V_1) \longrightarrow \pi(V_2(l))$$

for all l . The exts in the new category looks like,

$$\begin{aligned} Ext_{A_{\{x_0\}}}^1(V_i, V_j)^{k^*} &= 0, \text{ for } i=1, j=1,2. \\ Ext_{A_{\{x_0\}}}^1(V_2(l), V_1)^{k^*} &= k \text{ represented by } \xi = l \\ Ext_{A_{\{x_0\}}}^1(V_2(l), V_2(l'))^{k^*} &= k \text{ represented by } \xi = l \\ Ext_{A_{\{x_0\}}}^1(V_2(l), V_2(l'))^{k^*} &= 0 \text{ if } l \neq l' \end{aligned}$$

With this we find,

$$Spec(A_{\{x_0\}})/k^* = \mathcal{O}(Simp^*(A_{\{x_0\}} - k^*, \pi) = \left\{ \begin{pmatrix} f(x_1/x_0) & 0 \\ \psi(f(x_1/x_0))t_{2,1} & f(x_1/x_0) \end{pmatrix} \right\}$$

where ψ is some derivation of $Der_k(k[x_1/x_0])$ and f runs through $k[x_1/x_0]$ in,

$$\begin{pmatrix} End_k(k[x_1/x_0]) & 0 \\ k[[x_1/x_0]]t_{2,1} & k[[t_2(l)]] \end{pmatrix}$$

as expected, see the Theorem (4.2).

It is therefore clear that the non-commutative version of the projective line *contains* the geometric projective line.

If we consider, instead of the action by the group k^* , the action of the Lie algebra \mathfrak{g} generated by the Euler vectorfield $\delta_0 = x_0 \frac{\partial}{\partial x_0} + x_1 \frac{\partial}{\partial x_1}$, we get a different picture stemming from the fact that \mathfrak{g} has cohomology. The subcategory $Spec(A-\mathfrak{g})$ of $A-\mathfrak{g}$ -modules consists of the origin V_3 , the lines through the origin $V_2(l)$, and the generic point V_1 . The trivializing functor

$$\pi : A - \mathfrak{g} - mod \longrightarrow k - mod$$

has the values,

$$\pi(V_1) = k, \pi(V_2(l)) = k, \pi(V_3) = k$$

Since there are no π -incidences, the non-commutative orbit space $Spec(A)/\mathfrak{g}$ is given by the hull of the deformation functor, i.e. by $(H_{i,j})$, as above. However, here we cannot use the result (4.2), since for most \mathfrak{g} -modules V , $H^1(\mathfrak{g}, V) = V/\delta_0 V \neq 0$. In fact we get,

$$\begin{aligned} Ext_{A-\mathfrak{g}}^1(V_i, V_j) &= Ext_A^1(V_i, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_i, V_j)) \\ Ext_{A-\mathfrak{g}}^2(V_i, V_j) &= H^1(\mathfrak{g}, Ext_A^1(V_i, V_j)) \end{aligned}$$

This implies that

$$\begin{aligned}
Ext_{A-\mathfrak{g}}^1(V_1, V_j) &= H^1(\mathfrak{g}, Hom_A(V_1, V_j)) = k \text{ for } j=1,2,3. \\
Ext_{A-\mathfrak{g}}^1(V_2(l), V_j) &= Ext_A^1(V_2, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_2, V_j)) \\
&= k \oplus 0 \text{ for } j=1 \\
&= k \oplus k \text{ for } V_j = V_2(l) \\
&= 0 \oplus 0 \text{ for } V_j = V_2(l') \text{ } l \neq l' \\
&= 0 \oplus k \text{ for } j=3 \\
Ext_{A-\mathfrak{g}}^1(V_3, V_j) &= Ext_A^1(V_3, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, Hom_A(V_3, V_j)) \\
&= 0 \oplus 0 \text{ for } j=1 \\
&= 0 \oplus 0 \text{ for } j=2 \\
&= 0 \oplus k \text{ for } j=3 \\
Ext_{A-\mathfrak{g}}^2(V_i, V_j) &= H^1(\mathfrak{g}, Ext_A^1(V_i, V_j)) \\
&= 0 \text{ for } i=1, j=1,2,3. \\
&= k \text{ for } i=2, j=1 \\
&= k \text{ for } V_i = V_2(l), V_j = V_2(l) \\
&= 0 \text{ for } V_i = V_2(l), V_j = V_2(l'), \text{ } l \neq l' \\
&= k \text{ for } i=3, j=3
\end{aligned}$$

It follows that $Spec(A)/\mathfrak{g}$ is given by the rather complicated looking k -algebra, generated by,

$$\begin{pmatrix} k[[t_1]] & t_{1,2}(l) & t_{1,3}(l) \\ u_{2,1}(l) & k[[t_2(l), u_2(l)]] & t_{2,3}(l) \\ 0 & 0 & k[[t_3]] \end{pmatrix}$$

with some relations.

The moduli space of simple singularities, the A_2 case. We shall consider the Weierstrass family $F := F(t_0, t_1, x, y) = x^3 - y^2 + t_1x + t_0$, parametrized by the k -algebra $A := k[t_0, t_1]$, and the corresponding Kodaira-Spencer kernel $\mathfrak{g} \subseteq Der_k(A)$, generated by,

$$\begin{aligned}
\delta_0 &= 3t_0 \frac{\partial}{\partial t_0} + 2t_1 \frac{\partial}{\partial t_1} \\
\delta_1 &= 2t_1^2 \frac{\partial}{\partial t_0} - 9t_0 \frac{\partial}{\partial t_1}
\end{aligned}$$

We claim that the *moduli space* consisting of the three singularities in the family F , is given as the quotient space $Spec(A)/\mathfrak{g}$. We must therefore consider the diagram $Simp(A - \mathfrak{g})$, consisting of the 3 $A - \mathfrak{g}$ -modules, $V_1 = k[t_0, t_1]$, $V_2 = k[t_0, t_1]/(\Delta)$, where $\Delta = 27t_0^2 + 4t_1^3$ is the discriminant of F , and finally $V_3 = k$ corresponding to origo.

As above we find that

$$\pi = H^0(\mathfrak{g}, -) : A - \mathfrak{g} - mod \longrightarrow k - mod$$

defines three points,

$$\pi(V_1) = k, \pi(V_2) = k, \pi(V_3) = k$$

with no incidences. Since it is easy to see that

$$H^2(\mathfrak{g}, \text{Hom}_A(V_i, V_j)) = 0$$

we find,

$$\text{Ext}_{A-\mathfrak{g}}^1(V_i, V_j) = H^0(\mathfrak{g}, \text{Ext}_A^1(V_i, V_j)) \oplus H^1(\mathfrak{g}, \text{Hom}_A(V_i, V_j))$$

which implies,

$$\begin{aligned} \text{Ext}_{A-\mathfrak{g}}^1(V_1, V_j) &= H^1(\mathfrak{g}, \text{Hom}_A(V_1, V_j)) = k \text{ for } j=1,2,3. \\ \text{Ext}_{A-\mathfrak{g}}^1(V_2, V_j) &= H^0(\mathfrak{g}, \text{Ext}_A^1(V_2, V_j)) \oplus H^1(\mathfrak{g}, \text{Hom}_A(V_2, V_j)) \\ &= k \oplus 0 \text{ for } j=1 \\ &= 0 \oplus k \text{ for } j=2 \\ &= 0 \oplus k \text{ for } j=3 \\ \text{Ext}_{A-\mathfrak{g}}^1(V_3, V_j) &= \text{Ext}_A^1(V_3, V_j)^{\mathfrak{g}} \oplus H^1(\mathfrak{g}, \text{Hom}_A(V_3, V_j)) \\ &= 0 \oplus 0 \text{ for } j=1 \\ &= 0 \oplus 0 \text{ for } j=2 \\ &= 0 \oplus k \text{ for } j=3 \end{aligned}$$

Moreover,

$$\begin{aligned} \text{Ext}_{A-\mathfrak{g}}^2(V_i, V_j) &\subseteq H^0(\mathfrak{g}, \text{Ext}_A^2(V_i, V_j)) \oplus H^1(\mathfrak{g}, \text{Ext}_A^1(V_i, V_j)) \\ &= 0 \text{ for all } i,j=1,2,3. \end{aligned}$$

The moduli space is therefore given by the k -algebra freely generated by,

$$\begin{pmatrix} k[[t_{1,1}]] & t_{1,2} & t_{1,3} \\ 0 & k[[t_{2,2}]] & t_{2,3} \\ 0 & 0 & k[[t_{3,3}]] \end{pmatrix}$$

which has a *reduced* quotient, given by the matrices of the form,

$$\begin{pmatrix} k & kt_{1,2} & kt_{1,3} \oplus kt_{1,2}t_{2,3} \\ 0 & k & kt_{2,3} \\ 0 & 0 & k \end{pmatrix}$$

which is the k -algebra of the non-commuting adjacency diagram corresponding to the Weierstrass family, see [16],

$$\begin{aligned} t_{2,3} &: \text{cusp} \rightarrow \text{node} \\ t_{1,2} &: \text{node} \rightarrow \text{ellipt} \\ t_{1,2}t_{2,3} &: \text{cusp} \longrightarrow \text{ellipt} \\ t_{1,3} &: \text{cusp} \rightarrow \text{ellipt} \end{aligned}$$

Notice that \mathfrak{g} is a rank 2 A -module, such that we may expect to find exact sequences of $A - \mathfrak{g}$ -modules,

$$\begin{aligned} 0 &\longrightarrow A \longrightarrow \mathfrak{g} \longrightarrow A \longrightarrow 0 \\ 0 &\longrightarrow A/(\Delta) \longrightarrow \mathfrak{g} \otimes_A A/(\Delta) \longrightarrow A/(\Delta) \longrightarrow 0 \\ 0 &\longrightarrow A/(t_0, t_1) \longrightarrow \mathfrak{g} \otimes_A k(o) \longrightarrow A/(t_0, t_1) \longrightarrow 0 \end{aligned}$$

explaining the diagonal tangent structure of the quotient space,

$$\begin{pmatrix} k[[t_{1,1}]] & t_{1,2} & t_{1,3} \\ 0 & k[[t_{2,2}]] & t_{2,3} \\ 0 & 0 & k[[t_{3,3}]] \end{pmatrix}.$$

The moduli of endomorphisms. The dimension 2 case. We shall compute the (non-commutativ) space of invariants

$$End_k(k^n)/Gl_n(k)$$

for $k = \mathbf{C}$ and $n=2$. Referring to section (4) it suffices, to compute the non-commutative formal moduli for the longest chain of infinitesimal incidences. Let

$$A = O_{End_k(k^2)}(End_k(k^2))$$

then

$$A = k[x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}]$$

The group $G := Gl_2(k)$ acts on $End_k(k^2)$ by conjugation, and there are two Jordan formes of interest,

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

corresponding to orbits V_2 of dimension 0 and V_1 of dimension 2, respectively, in $End_k(k^2)$. According to section (4), since

$$H^0(G, V_i) = k, \quad i = 1, 2$$

we should expect $End_k(k^2)/Gl_2(k)$ to be an algebraization of the formal moduli of the family of A - G -modules $\{V_i\}_{i=1,2}$, i.e.

$$H := H(\{V_i\}_{i=1,2})$$

The orbits, as A -modules, are given by:

$$\begin{aligned} V_1 &= k[x_{i,j}]/(s_1 - 2\lambda, s_2 - \lambda^2) \\ V_2 &= k[x_{i,j}]/(x_{1,1} - \lambda, x_{1,2}, x_{2,1}, x_{2,2} - \lambda) \end{aligned}$$

Now it is clear that we may assume $\lambda = 0$. We find A-free resolutions,

$$\begin{array}{ccccccc}
 V_1 & \longleftarrow & A & \xleftarrow{(s_1, s_2)} & A^2 & \xleftarrow{\begin{pmatrix} s_2 \\ s_1 \end{pmatrix}} & A \longleftarrow 0 \\
 & & \searrow \psi(1)_1 & & \searrow \psi(1)_2 & & \\
 V_1 & \longleftarrow & A & \xleftarrow{\psi(2)_1} & A^2 & \xleftarrow{\psi(2)_2} & A \longleftarrow 0 \\
 & & \searrow \psi_{1,2}(1)_1 & & \searrow \psi_{1,2}(1)_2 & & \\
 V_2 & \longleftarrow & A & \xleftarrow{\rho} & A^4 & \xleftarrow{d_1} & A^6 \longleftarrow A^4 \\
 & & \searrow \psi_1 & & \searrow \psi_2 & & \\
 V_2 & \longleftarrow & A & \xleftarrow{\rho} & A^4 & \xleftarrow{d_1} & A^6 \longleftarrow A^4
 \end{array}$$

Here,

$$\begin{aligned}
 \psi(1)_1 &= (1, 0) \\
 \psi(2)_1 &= (0, 1) \\
 \psi(1)_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 \psi(2)_2 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
 \psi_{(1,2)}(1)_1 &= (1, 0) \\
 \psi_{(1,2)}(2)_1 &= (0, 1) \\
 \psi_{(1,2)}(1)_2 &= \begin{pmatrix} x_{2,2} \\ -x_{2,1} \\ 0 \\ 0 \end{pmatrix} \\
 \psi_{(1,2)}(2)_2 &= \begin{pmatrix} -1 \\ 0 \\ 0 \\ -1 \end{pmatrix}
 \end{aligned}$$

and

$$\begin{aligned}
 \rho &= (x_{1,1} \quad x_{1,2} \quad x_{2,1} \quad x_{2,2}) \\
 d_1 &= \begin{pmatrix} x_{1,2} & x_{2,1} & x_{2,2} & 0 & 0 & 0 \\ -x_{1,1} & 0 & 0 & x_{2,1} & x_{2,2} & 0 \\ 0 & -x_{1,1} & 0 & -x_{1,2} & 0 & x_{2,2} \\ 0 & 0 & -x_{1,1} & 0 & -x_{1,2} & -x_{2,1} \end{pmatrix} \\
 \psi_1 &= (1 \quad 0 \quad 0 \quad 1) \\
 \psi_2 &= \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}
 \end{aligned}$$

From this diagram we easily compute $Ext_A^1(V_i, V_j)$, and since G is reductive, also $Ext_{A-G}^1(V_i, V_j) = Ext_A^1(V_i, V_j)^G$. We obtain,

$$Ext_{A-G}^1(V_i, V_j) = \begin{cases} = k^2 & \text{for } i = 1 \\ = k & \text{for } i = j = 2 \\ = 0 & \text{for } i = 2, j = 1 \end{cases}$$

which means that the tangent space of H is given by,

$$\begin{pmatrix} k^2 & k^2 \\ 0 & k^1 \end{pmatrix}.$$

Now,

$$\begin{aligned} Ext_{A-G}^2(V_1, V_1) &= k \cdot \eta_{1,1} \\ Ext_{A-G}^2(V_1, V_2) &= k \cdot \eta_{1,2} \end{aligned}$$

and we compute the cup products and the Massey products of the basis elements of the $Ext_{A-G}^1(V_i, V_j) = Ext_A^1(V_i, V_j)^G$,

$$\begin{aligned} s_1^* \cup s_2^* &= -s_2^* \cup s_1^* = \eta_{1,1}, s^* \cup s^* = 0 \\ t_1^* \cup s^* &= 0, t_2^* \cup s^* = -2 \cdot \eta_{1,2}, \\ < t_1^*, s^*, s^* > &= 1/2 \cdot \eta_{1,2} \\ s_1^* \cup t_1^* &= 0, s_1^* \cup t_2^* = \eta_{1,2}, s_2^* \cup t_1^* = -\eta_{1,2}, s_2^* \cup t_2^* = 0, \end{aligned}$$

This proves that in H there are relations of the form,

$$(rel) \quad s_1 s_2 = s_2 s_1, \quad 1/2 \cdot t_1 s^2 - 2 \cdot t_2 s + s_1 t_2 - s_2 t_1 = 0$$

Dividing the generators s_1 and s_2 by 2, keeping the notations, it follows that,

$$H(\{V_i\}) = \begin{pmatrix} k[[s_1, s_2]] & < t_1, t_2 > \\ 0 & k[[s]] \end{pmatrix}$$

subject to the relation $t_1 s^2 - 4 \cdot t_2 s + 4 \cdot s_1 t_2 - 4 \cdot s_2 t_1 = 0$. From this it is clear that,

$$End_k(k^n)/Gl_n(k) = \tilde{H}(\{V_i\}) = \begin{pmatrix} k[s_1, s_2] & < t_1, t_2 > \\ 0 & k[s] \end{pmatrix}$$

subject to the relation $t_1 s^2 - 4 \cdot t_2 s + 4 \cdot s_1 t_2 - 4 \cdot s_2 t_1 = 0$.

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